Consumer Choice 1

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GR5211 - Microeconomic Analysis 1

The Indirect Utility Function

- Imagine that the consumer can choose to live in two different countries
  - In country 1 they would face prices p^1 and have income w^1
  - In country 2 they would face prices p^2 and have income w^2
- Which country would they prefer to live in?
- i.e. what are there preferences over budget sets?
  - which we can denote by $\succeq^*$

- Here is one possibility
  - Figure out one of the best items in budget set 1 (i.e. $x(p^1, w^1)$)
  - Figure out one of the best items in budget set 2 (i.e. $x(p^2, w^2)$)
  - The consumer prefers budget set 1 to budget set 2 if the former is preferred to the latter
- i.e. we can define $\succeq^*$on the set of budget sets by

$$\begin{aligned} (p^1, x^1) &\succeq {}^*(p^2, x^2) \\ \text{if and only if } x^1 &\succeq x^2 \\ \text{for } x^1 &\in x(p^1, x^1) \text{ and } x^2 \in x(p^2, x^2) \end{aligned}$$

- Can you think of reasons why this might not be the right model?
  - Temptation
  - Uncertainty
  - Regret

The Indirect Utility Function

- If $\succeq$ can be represented by a utility function we can define the **indirect utility function**

$$v(p, w) = u(x(p, w))$$

- v now represents the preferences $\succeq^*$ on the space of budget sets
  - Proof?

Properties of the Indirect Utility Function

- **Property 1:**

$$v(\alpha p, \alpha w) = v(p, w) \text{ for } \alpha > 0$$

  - Follows from the fact that $x(\alpha p, \alpha w) = x(p, w)$
- **Property 2:** $v(p, w)$ is non increasing in p and increasing in w
  - Assuming non satiation

Properties of the Indirect Utility Function

- **Property 3:** v is quasiconvex: i.e. the set

$$\{(p, w) | v(p, w) \leq \bar{v}\}$$

  is convex for all $\bar{v}$
  - Proof left as an exercise
- **Property 4:** If $\succeq$ is continuous then $\succeq^*$ is continuous
  - Follows from the Theorem of the Maximum

The Story of The Turtle

- From Ariel Rubinstein
  - The furthest a turtle can travel in 1 day is 1 km
  - The shortest length of time it takes for a turtle to travel 1km is 1 day
- No, we didn't know what he was on about either
- But bear with me...

- Is this always true?
- No! Requires two assumptions
  1. The turtle can travel a strictly positive distance in any positive period of time
  2. The turtle cannot jump a positive distance in zero time
- So much for zoology, what has this got to do with economics?

- It is going to be very useful to define **Expenditure minimization problem**
  - This is the **dual** of the utility maximization problem
- **Prime problem** (utility maximization)

$$\begin{aligned} \text{choose } x &\in \mathbb{R}^N_+ \\ &\text{in order to maximize } u(x) \\ \text{subject to } \sum_{i=1}^{N} p_i x_i &\leq w \end{aligned}$$

- **Dual problem** (cost minimization)

$$\begin{aligned} \text{choose } x &\in \mathbb{R}^N_+ \\ &\text{in order to minimize } \sum_{i=1}^{N} p_i x_i \\ \text{subject to } u(x) &\geq \bar{u} \end{aligned}$$

- Are these problems ‘the same’?
- In general, no
  - Like the teleporting turtle
- However, if we rule out teleportation (and laziness) then they will be the same.
- What assumptions allow us to do that?

Theorem

If u is monotonic and continuous then x^* is a solution to the prime problem with prices p and wealth w it is a solution to the dual problem with prices p and utility $v(p, w)$

Proof.

- Assume not, then there exists a bundle $\bar{x}$ such that

$$u(\bar{x}) \geq v(p, w) = u(x^*)$$

with

$$\sum p_i \bar{x}_i < \sum p_i x_i^* = w$$

- But this means, by monotonicity, that there exists an $\varepsilon > 0$ such that, for

$$x' = \begin{pmatrix} \bar{x}_1 + \varepsilon \\ \bar{x}_2 + \varepsilon \\ \vdots \\ \bar{x}_N + \varepsilon \end{pmatrix}$$

$$\sum p_i x_i' < w$$

□

Proof.

- By monotonicity, we know that $u(x') > u(\bar{x}) \geq u(x^*)$, and so x^* is not a solution to the prime problem

□

Theorem

If u is monotonic and continuous then x^* is a solution to the dual problem with prices p and utility u^* it is a solution to the prime problem with prices p and wealth $\sum p_i x_i^*$

Proof.

- Assume not, then there exists a bundle $\bar{x}$ such that

$$\sum p_i \bar{x}_i \leq \sum p_i x_i^*$$

with

$$u(\bar{x}) > u(x^*) \geq u^*$$

- By continuity, there exists an $\varepsilon > 0$ such that, for all $x' \in B(\bar{x}, \varepsilon)$, $u(x') > u(x^*)$

□

Proof.

- In particular, there is an $\varepsilon > 0$ such that

$$x' = \begin{pmatrix} \bar{x}_1 - \varepsilon \\ \bar{x}_2 - \varepsilon \\ \vdots \\ \bar{x}_N - \varepsilon \end{pmatrix}$$

and $u(x') > u(x^*) \geq u^*$

- But $\sum p_i x_i' < \sum p_i \bar{x}_i \leq \sum p_i x_i^*$, so x^* is not a solution to the prime problem.

□

Hicksian Demand and the Expenditure Function

- The dual problem allows us to define two new objects
- The **Hicksian demand function**

$$
\begin{aligned}
h(p,u) &= \arg\min_{x \in X} \sum p_i x_i \\
\text{subject to } u(x) &\geq \bar{u}
\end{aligned}
$$

  - This is the demand for each good when prices are p and the consumer must achieve utility u
  - Note difference from Walrasian demand

- The **expenditure function**

$$
\begin{aligned}
e(p,u) &= \min_{x \in X} \sum p_i x_i \\
\text{subject to } u(x) &\geq \bar{u}
\end{aligned}
$$

  - This is the amount of money necessary to achieve utility u when prices are p

Properties of the Hicksian Demand Function

- Assume that we are dealing with continuous, non-satiated preferences
- **Fact 1**: h is homogenous of degree zero in prices - i.e. $h(\alpha p, u) = h(p, u)$ for $\alpha > 0$
  - Follows from the fact that increasing all prices by α does not change the tangency conditions
  - i.e. the slope of the 'budget line' remains the same
- **Fact 2:** No excess utility - i.e. $u(h(p, u)) = u$
  - Follows from continuity (why?)

Properties of the Hicksian Demand Function

- **Fact 3**: If preferences are convex then h is a convex set. If preferences are strictly convex then h is unique
  - Proof: say that x and y are both in $h(p,u)$. Then

$$\sum p_i x_i = \sum p_i y_i = e(p,u)$$

  - Implies that for any $\alpha \in (0,1)$ and $z = \alpha x + (1-\alpha)y$

$$\begin{aligned}\sum p_i z_i &= \sum p_i \left(\alpha x_i + (1-\alpha) y_i\right) \\ &= \alpha \sum p_i x_i + (1-\alpha) \sum p_i y_i \\ &= e(p,u)\end{aligned}$$

  - Also, as preferences are convex, $z \succeq x$, and so $u(z) \geq u(x) = u$
  - If preferences are **strictly** convex, then $z \succ x$
  - But, by continuity, exists $\varepsilon > 0$ such that $z' \succ x$ all $z' \in B(x, \varepsilon)$
  - Implies that there is a z' such that $u(z') > u$ and $\sum p_i z_i < \sum p_i x_i$

Properties of the Expenditure Function

- Again, assume that we are dealing with continuous, non-satiated preferences
- **Fact 1:** $e(\alpha p, u) = \alpha e(p, u)$
  - Follows from the fact that $h(\alpha p, u) = h(p, u)$
- **Fact 2:** e is strictly increasing in u and non-decreasing in p
  - Strictly increasing due to continuity and non-satiation
  - Only non-decreasing because may already be buying 0 of some good
- **Fact 3:** e is continuous in p and u
  - Logic follows from the theorem of the maximum (though can't be applied directly)

Properties of the Expenditure Function

- **Fact 4:** e is concave in p
  - Proof: fix a $\bar{u}$. we need to show that

$$e(p'', \bar{u}) \geq \alpha e(p, \bar{u}) + (1-\alpha) e(p', \bar{u})$$

where

$$p'' = \alpha p + (1-\alpha) p'$$

  - Let $x'' \in h(p'', \bar{u})$, then

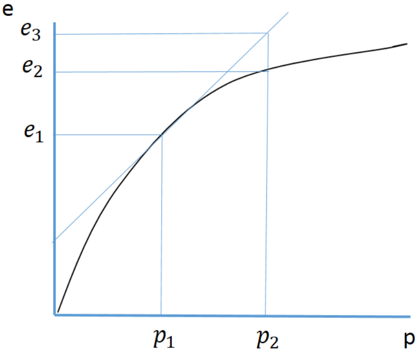
$$\begin{aligned} e(p'', \bar{u}) &= \sum p_i'' x_i'' \\ &= \sum \left(\alpha p_i + (1-\alpha) p_i'\right) x_i'' \\ &= \alpha \sum p_i x_i'' + (1-\alpha) \sum p_i' x_i'' \\ &\geq \alpha \sum p_i x_i + (1-\alpha) \sum p_i' x_i' \\ &= \alpha e(p, \bar{u}) + (1-\alpha) e'(p, \bar{u}) \end{aligned}$$

where $x \in h(p, \bar{u})$ and $x' \in h(p', \bar{u})$

Properties of the Expenditure Function

- This is quite an important and intuitive property
- Implies that if we look at how expenditure changes as a function of one price it looks like this ...

Properties of the Expenditure Function



- Think of a price increase from p_1 to p_2
- If the consumer couldn't change their allocation then expenditure would go from e_1 to e_3
- This is an upper bound on the true increase in expenditure.

- We will now put the above machinery to work to learn about the relationship between the various measures we have introduced
- This will also allow us to say something about the comparative statics of these functions - for example how demand changes with price
- Before doing so, it will be worth reviewing a very useful mathematical result
    - The Envelope Theorem
    - See Mas-Colell section M.L

- Consider a constrained optimization problem

$$\begin{aligned} & \text{choose } x \\ \text{in order to maximize } f(x &: q) \\ & \text{subject to} \\ g_1(x &: q) = 0 \\ & \vdots \\ g_N(x &: q) = 0 \end{aligned}$$

- Where q are some parameters of the problem (for example prices)

- Assume the problem is well behaved, and let
  - $x(q)$ be (a) solution to the problem if the parameters are q
  - $v(q) = f(x(q) : q)$
- Key question: how does v alter with q
  - i.e. how does the value that can be achieved vary with the parameters?

The Envelope Theorem

- Say that both x and q are single valued
- And say that there are no constraints
- Chain rule gives

$$\frac{\partial v}{\partial q} = \frac{\partial f}{\partial q} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial q}$$

- But note that if we are at a maximum

$$\frac{\partial f}{\partial x} = 0$$

  - and so

$$\frac{\partial v}{\partial q} = \frac{\partial f}{\partial q}$$

- **Only the direct effect of the change in parameters matters**

- This result generalizes

Theorem (The Envelope Theorem)

In the above decision problem

$$\frac{\partial v(\bar{q})}{\partial q} = \frac{\partial f(x(\bar{q}) : \bar{q})}{\partial q} - \sum_n \lambda_n \frac{\partial g_n(x(\bar{q}) : \bar{q})}{\partial q}$$

where λ_n is the Lagrange multiplier on the nth constraint

- We can now apply the envelope theorem to get some interesting results relating the various functions that we have defined
- First, the relationship between the expenditure function and Hicksian demand

Theorem (Shephard's Lemma)

Say preferences are continuous, locally non satiated and strictly convex then

$$h_l(p,u) = \frac{\partial e(p,u)}{\partial p_l}$$

Proof.

EMP is

$$\min \sum_{i=1}^{N} p_i x_i$$
$$\text{subject to } u(x) \geq u$$

Applying the envelope theorem directly gives the result □

Hicksian Demand and The Expenditure Function

Corollary

Assume h is continuously differentiable, and let

$$D_p h(p,u) = \begin{pmatrix} \frac{\partial h_1}{\partial p_1} & \cdots & \frac{\partial h_1}{\partial p_M} \\ \vdots & & \vdots \\ \frac{\partial h_M}{\partial p_1} & \cdots & \frac{\partial h_M}{\partial p_M} \end{pmatrix}$$

Then

1. $D_P h(p,u) = D_p^2 e(p,u)$
2. $D_P h(p,u)$ *is negative semi definite*
3. $D_P h(p,u)$ *is symmetric*
4. $D_P h(p,u) p = 0$

Proof.

1. Follows directly from previous claim
2. Follows from (1) and the fact that e is concave
3. Follows from (1) and the fact that matrices of second derivatives are symmetric
4. Follows from the homogeneity of degree zero of h, so

$$h(\alpha p, u) - h(p, u) = 0$$

Differentiating with respect to α gives the desired result

□

Theorem (Roy's Identity)

Say preferences are continuous, locally non satiated and strictly convex then

$$x_l(p, w) = -\frac{\frac{\partial v(p,w)}{\partial p_l}}{\frac{\partial v(p,w)}{\partial w}}$$

Proof.

Applying the envelope theorem tells us that

$$\frac{\partial v(p, w)}{\partial p_l} = -\lambda x_l(p, w)$$

also

$$\lambda = \frac{\partial v(p, w)}{\partial w}$$

□

- Perhaps more usefully we can relate Hicksian and Walrasian Demand

Theorem (The Slusky Equation)

Let preferences be continuous, strictly convex and locally non-satiated and $u = v(p, w)$

$$\frac{\partial h_l(p,u)}{\partial p_k} = \frac{\partial x_l(p,w)}{\partial p_k} + \frac{\partial x_l(p,w)}{\partial w} x_k(p_w, w)$$

- Proof.

  By duality, we know

$$h_l(p,u) = x_l(p, e(p,u))$$

Differentiating both sides with respect to p_k gives

$$\frac{\partial h_l(p,u)}{\partial p_k} = \frac{\partial x_l(p,w)}{\partial p_k} + \frac{\partial x_l(p,w)}{\partial w}\frac{\partial e(p,u)}{\partial p_k}$$

but we know that

$$\frac{\partial e(p,u)}{\partial p_k} = h_k(p,u) = x_k(p, e(p,u)) = x_k(p,w)$$

□

- Why is this useful?
- Define the Slutsky Matrix by

$$S_{l,,k} = \frac{\partial x_l(p,w)}{\partial p_k} + \frac{\partial x_l(p,w)}{\partial w} x_k(p,w)$$

- The above theorem tells us that

$$S = D_P h(p,u)$$

- And so S must be negatively semi definite, symmetric and $S.p = 0$
- Also note that S is observable (if you know the demand function)
- It turns out this result is if and only if: Demand is rationalizable if and only if the resulting Slutsky Matrix has the above properties

- It also helps us understand how demand changes as respond to own prices.
- We now need one more theorem

Theorem (The Law of Compensated Demand)

Assume preferences are continuous, locally non satiated and strictly convex, then for any p', p''

$$(p'' - p')(h(p'', u) - h(p', u) \leq 0$$

Proof.

As h minimizes expenditure we have

$$p''h(p'', u) \leq p''h(p', u)$$

and

$$p'h(p'', u) \geq p'h(p', u)$$

Subtracting the two inequalities gives the result □

Law of Compensated Demand

- An immediate corollary is that the compensated price elasticity of demand is non positive
  - An increase in the price of good l reduces the Hicksian demand for good l
- Back to the Slutsky equation we $l = k$ we have

$$\frac{\partial h_l(p,u)}{\partial p_l} - \frac{\partial x_l(p,w)}{\partial w} x_l(p,w) = \frac{\partial x_l(p,w)}{\partial p_l}$$

- Does $\frac{\partial x_l(p,w)}{\partial p_l}$ have to be negative?
  - No! Giffen Goods
- But this can only happen if the income effect

$$\frac{\partial x_l(p,w)}{\partial w} x_l(p,w)$$

- Overwhelms the substitution effect

$$\frac{\partial h_l(p,u)}{\partial p_l}$$