A Representation Theorem for Utility Maximization

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GR5211 - Microeconomic Analysis 1

A Representation Theorem

- When dealing with models that have latent (or unobservable) variables (such as utility maximization) we will want to find a *representation theorem*
- This consists of three things
  - A data set
  - A model
  - A set of conditions on the data which are **necessary** and **sufficient** for it to be consistent with the model
- Means testing these conditions is the same as testing the model itself

A Representation Theorem for Utility Maximization

- We are now going to develop a representation theorem for the model of utility maximization
- This is largely just formalizing the intuition we developed on the previous slides
- It is going to lead us to introduce a new model - that of preference maximization.

- The data we are going to use are the choices people make
- Notation:
  - X: Set of objects you might get to choose from: **to begin with we will assume this is finite**
  - 2^X: The power set of X (i.e. all the subsets of X)
  - $\emptyset$: The empty set
- Our data is going to take the form of a **choice correspondence** which tells us what the person chose from each subset of X

Definition

A choice correspondence C is a mapping $C : 2^X/\emptyset \rightarrow 2^X/\emptyset$ such that $C(A) \subset A$ for all $A \in 2^X/\emptyset$.

Notes

- This is just a way of recording what we described previously
- For example, if we offered someone the choice of Jaffa Cakes and Kit Kats, and they chose Jaffa Cakes, we would write

$$C(\{kitkat, jaffacakes\}) = \{jaffacakes\}$$

- C is just a record of the choices made from all possible choice sets
  - i.e. all sets in 2^X apart from the empty set $\emptyset$
- We insist that the DM chooses something that was actually in the data set
  - i.e. $C(A) \subset A$
- **Important**: Choice correspondence is non-empty: something is chosen from each choice set

Notes

- What are some issues with this data set?

1. X Finite
2. Observe choices from all choice sets
3. We allow for people to choose more than one option!
    - i.e. we allow for data of the form

$$C(\{kitkat, jaffacakes, lays\}) = \{jaffacakes, kitkat\}$$

    - Which we interpret as something like "the decision maker would be happy with either jaffa cakes or lays from this choice set"

- We will think about all of these issues later on, but let's start simple!

Utility Maximization

- The model we want to test is that of utility maximization
- i.e. there exists a utility function $u : X \rightarrow \mathbb{R}$
- Such that the things that are chosen are those which maximize utility
  - For every A
$$C(A) = \arg\max_{x \in A} u(x)$$
- If this is true, we say that u **rationalizes** C
- If C can be rationalized by some u then we say it has a **utility representation**

- We want to know when data is consistent with utility maximization
  - i.e. it has a utility representation
- So we would like to find a set of conditions on C such that it has a utility representation **if and only if** these conditions are satisfied
  - Testing these conditions is then the same as testing the model of utility maximization

- You may remember a condition called the Weak Axiom of Revealed Preference from Intermediate Micro
- We will break WARP down into two parts

Axiom α (AKA Independence of Irrelevant Alternatives) If $x \in B \subseteq A$ and $x \in C(A)$, then $x \in C(B)$

Axiom β If $x, y \in C(A)$, $A \subseteq B$ and $y \in C(B)$ then $x \in C(B)$

- Notice we can **test** these conditions!
- If we have data, we can see if they are satisfied

- These conditions form the basis of our first representation theorem

Theorem

A Choice Correspondence on a finite X has a utility representation **if and only if** *it satisfies axioms α and β*

- **if**: if α and β are satisfied then a utility representation exists
- **only if:** if a utility representation exists then α and β are satisfied

- We are going to prove this theorem
- Before we do so, we are going to introduce the notion of **preferences**, and the associated model of **preference maximization**
  - These are going to be useful when proving our theorem
  - Also of independent interest as a model of choice

The Questionnaire

- Consider the alternatives in X
  - e.g. Jaffa cakes, Kit kat, Lays
- Consider an exhaustive list of questions:

  Do you like alternative x as much as alternative y?

- If the answer is yes, then we write $x \succeq y$
- Technically speaking, this is a **binary relation**
  - A subset of $X \times X$ which contains all x, y such that $x \succeq y$
- Where do these preferences come from?
  - Could be choices (we will come back to this)
  - But we could ask people to express preferences over objects that we couldn't actually give them....

The Questionnaire

- Should we allow any possible answers to the questionnaire?
- No! Or at least we are going to rule some things out.
  - You cannot answer 'I don't know' or 'I like x much more than y' (only yes or no answers)
  - You have to answer 'yes' either to the question
    - Do you like alternative x as much as alternative y?
  - or
    - Do you like alternative y as much as alternative x?
  - Coherence
    - If you like x as much as y and y as much as z you must say that you like x as much as z
- Do these seem like sensible properties?

- This means that the binary relation $\succeq$ has certain properties
  - Completeness: for every x and y in X either $x \succeq y$ or $y \succeq x$ (or both)
  - Transitivity: if $x \succeq y$ and $y \succeq z$ then $x \succeq z$
  - Reflexive: $x \succeq x$
- Such binary relations are called **complete preference relations** or a **complete preorder**
- Does it have other properties (if not, can you think of binary relations that do)?
  - Antisymmetric: $xRyRx$ implies $x = y$
  - Asymmetric: If xRy then not yRx
  - Symmetry: xRy implies yRx

- Notice that we can use $\succeq$ to define other binary relations:
  - Strict Preference

$$x \succ y : \text{ if } x \succeq y \text{ but not } y \succeq x$$

  - Indifference

$$x \sim y : \text{if } x \succeq y \text{ and } y \succeq x$$

- What properties do these binary relations have?

Preference Relations and Choice

- We can use preferences to form a model of choice
- We say that the complete preference relation $\succeq$ represents a choice function C if, for every A

$$C(A) = \{x \in A | x \succeq y \text{ for all } y \in A\}$$

  - i.e. the things that are chosen are those that are preferred to everything else in the choice set

- We can also think about relating preferences and utility.
  - i.e. we can treat preferences as **data**
- We say that a utility function u **represents** preferences $\succeq$ if

$$\begin{aligned} u(x) &\geq u(y) \text{ if and only if} \\ x &\succeq y \end{aligned}$$

- In fact, this is how we are going to prove our representation theorem
- If we can find
  - A preference relation which represents choices
  - A utility function which represents preferences

  we are done!
- Preferences represents choices means

$$C(A) = \{x \in A | x \succeq y \text{ for all } y \in A\}$$

- Utility represents preferences means

$$u(x) \geq u(y) \Longleftrightarrow x \succeq y$$

- So

$$\begin{aligned} C(A) &= \{x \in A | u(x) \geq u(y) \text{ for all } y \in A\} \\ &= \arg\max_{x \in A} u(x) \end{aligned}$$

- Thus, in order to prove that axioms α and β are equivalent to utility maximization we will do the following

1. Show that if the data satisfies α and β then we can find a complete, transitive, reflexive preference relation $\succeq$ which represents the data
2. Show that if the preferences are complete, transitive and reflexive then we can find a utility function u which represents them
3. Show that if the data has a utility representation then it must satisfy α and β

- We will do 1 and 2 in class. You can do 3 for homework