Utility Maximization 2: Extensions

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- We have now proved the following theorem

Theorem

A Choice Correspondence on a finite X has a utility representation **if and only if** *it satisfies axioms α and β*

- Great! We know how to test the model of utility maximization!
- However, our theorem is only as useful as the data set we are working with
- As discussed at the time, there are some problems with the data we have assumed so far

Problems with the Data Set

- What are some issues with this data set?

1. Observe choices from all choice sets
2. We allow for people to choose more than one option!
   - i.e. we allow for data of the form

$$C(\{kitkat, jaffacakes, lays\}) = \{jaffacakes, kitkat\}$$

3. X Finite

Choices from all Choice Sets?

- Imagine running an experiment to try and test α and β
- The data that we need is the choice correspondence

$$C : 2^X / \emptyset \rightarrow 2^X / \emptyset$$

- How many choices would we have to observe?
- Lets say $|X| = 10$
  - Need to observe choices from every $A \in 2^X / \emptyset$
  - How big is the power set of X?
  - If $|X| = 10$ need to observe 1024 choices
  - If $|X| = 20$ need to observe 1048576 choices
- This is not going to work!

Choices from all Choice Sets?

- So how about we forget about the requirement that we observe choices from all choice sets
- Are α and β still enough to guarantee a utility representation?

$$\begin{aligned} C(\{x, y\}) &= \{x\} \\ C(\{y, z\}) &= \{y\} \\ C(\{x, z\}) &= \{z\} \end{aligned}$$

- If this is our only data then there is no violation of α or β
- But no utility representation exists!
- We need a different approach!

- In order to do this we are going to have to know a few more things about order theory (the study of binary relations)
- In particular we are going to need some definitions

Definition

A transitive closure of a binary relation R is a binary relation $T(R)$ that is the smallest transitive binary relation that contains R.

- i.e. $T(R)$ is
    - Transitive
    - Contains R in the sense that xRy implies $xT(R)y$
    - Any binary relation that is smaller (in the subset sense) is either intransitive or does not contain R
- Example?

- We can alternatively define the transitive closure of a binary relation R on X as the following:

Remark

- 1. *Define* $R_0 = R$
  2. *Define* R_m *as* xR_my *if there exists* $z_1, ..., z_m \in X$ *such that* $xRz_1R...Rz_mRy$
  3. $T = R \cup_{i \in \mathbb{N}} R_m$

Definition

Let $\succeq$ be a preorder on X. An **extension** of $\succeq$ is a preorder $\trianglerighteq$ such that

$$\begin{aligned} \succeq \quad &\subset \trianglerighteq \\ \succ \quad &\subset \triangleright \end{aligned}$$

Where

- $\succ$ is the asymmetric part of $\succeq$, so $x \succ y$ if $x \succeq y$ but not $y \succeq x$
- $\triangleright$ is the asymmetric part of $\trianglerighteq$, so $x \triangleright y$ if $x \trianglerighteq y$ but not $y \trianglerighteq x$

- Example?

- We are also going to need one theorem

Theorem (Sziplrajn)

For any nonempty set X and preorder $\succeq$ on X there exists a complete preorder that is an extension of $\succeq$

- Relatively easy to prove if X is finite, but also true for any arbitrary X

- Okay, back to choice
- The approach we are going to take is as follows:
  - Imagine that the model of preference maximization is correct
  - What observations in our data would lead us to conclude that x was preferred to y?

- We say that x is **directly revealed preferred to** y $(xR^D y)$ if, for some choice set A

$$\begin{aligned} y &\in A \\ x &\in C(A) \end{aligned}$$

- We say that x is **revealed preferred to** y (xRy) if we can find a set of alternatives w_1, $w_2, \ldots.w_n$ such that
  - x is directly revealed preferred to w_1
  - w_1 is directly revealed preferred to w_2
  - ...
  - w_{n-1} is directly revealed preferred to w_n
  - w_n is directly revealed preferred to y
- I.e. R is the transitive closure of R^D

- We say x is **strictly revealed preferred to** y (xSy) if, for some choice set A

$$\begin{aligned} y &\in A \textbf{ but not } y \in C(A) \\ x &\in C(A) \end{aligned}$$

Notes

- Is it always true that choosing x over y means that you prefer x to y?
- Almost certainly not
  - Think of a model of 'consideration sets'
- Only true in the context of the model of preference maximization

The Generalized Axiom of Revealed Preference

- Note that we can observe revealed preference and strict revealed preference from the data
- With these definitions we can write an axiom to replace α and β
- What behavior is ruled out by utility maximization?

Definition

A choice correspondence C satisfies the Generalized Axiom of Revealed Preference (GARP) if it is never the case that x is revealed preferred to y, and y is **strictly** revealed preferred to x

- i.e. xRy implies not ySx

The Generalized Axiom of Revealed Preference

Theorem

A choice correspondence C on an arbitrary subset of $2^X / \oslash$ satisfies GARP if and only if it has a preference representation

Corollary

A choice correspondence C on an arbitrary subset of $2^X / \oslash$ with X finite satisfies GARP if and only if it has a preference representation

Choices from all Choice Sets?

- Note that this data set violates GARP

$$\begin{aligned} C(\{x,y\}) &= \{x\} \\ C(\{y,z\}) &= \{y\} \\ C(\{x,z\}) &= \{z\} \end{aligned}$$

- xR^Dy and yR^Dz so xRz
- But zSx

The Generalized Axiom of Revealed Preference

- **Proof: GARP implies representation**
- First, note that R is transitive (and without loss of generality we can assume it is reflexive)
- Also note that, by GARP, S is the asymmetric part of R
- This means that, by Sziplrajn's theorem there exists a complete preference relation $\succeq$ such that

$$xRy \text{ implies } x \succeq y$$
$$xSy \text{ implies } x \succ y$$

The Generalized Axiom of Revealed Preference

- All we need to show is that $\succeq$ represents choice, i.e

$$C(A) = \{x \in A | x \succeq y \text{ all } y \in A\}$$

- Again, need to show two things
  1. $x \in C(A) \Rightarrow x \succeq y$ all $y \in A$
     - This follows from the fact that $x \in C(A) \Rightarrow xR^D y \;\forall\; y \in A$ and so $x \succeq y \;\forall\; y \in A$
  2. $x \in A$ and $x \succeq y$ all $y \in A \Rightarrow x \in C(A)$
     - Assume by way of contradiction $x \notin C(A)$, and take $y \in C(A)$
     - This implies that ySx and so $y \succ x$ and therefore not $x \succeq y$
     - Contradiction