Utility Maximization 2: Extensions

Mark Dean

GR5211 - Microeconomic Analysis 1

Choice Correspondence?

- Another weird thing about our data is that we assumed we could observe a choice **correspondence**
  - Multiple alternatives can be chosen in each choice problem
- This is not an easy thing to do!
- What about if we only get to observe a choice function?
  - Only one option chosen in each choice problem
- How do we deal with indifference?

Choice Correspondence?

- One of the things we could do is assume that the decision maker chooses **one of** the best options

$$C(A) \in \arg\max_{x \in A} u(x)$$

- Is this going to work?
- No!
- Any data set can be represented by this model
  - Why?
  - We can just assume that all alternatives have the same utility!

Choice Correspondence?

- Another thing we can do is assume away indifference

$$C(A) = \arg\max_{x \in A} u(x)$$

  - for some one-to-one function u
- Is this going to work?
- Yes
  - Implies that data is a function
  - Property α (or GARP) will be necessary and sufficient (if X is finite)
- But maybe we don't **want** to rule out indifference!
  - Maybe people are sometimes indifferent!

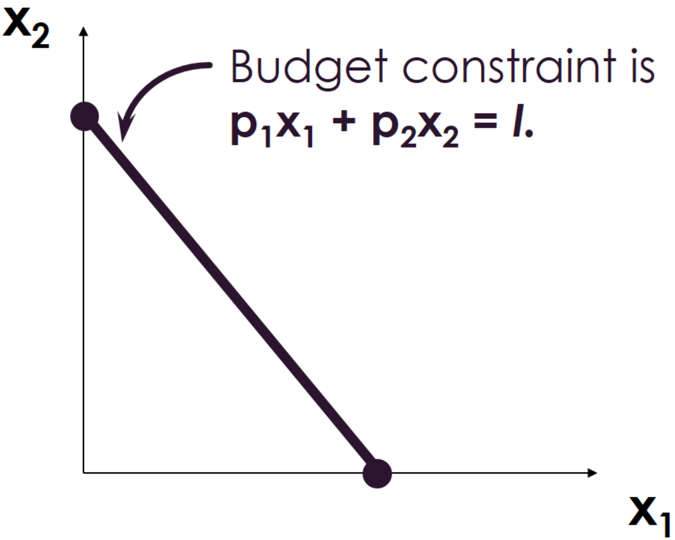
Choice from Budget Sets

- Need some way of identifying when an alternative x is **better than** alternative y
  - i.e. some way to identify strict preference
- One case in which we can do this is if our data comes from people choosing **consumption bundles** from **budget sets**
  - Should be familiar from previous economics courses
- The objects that the DM has to choose between are bundles of different commodities

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

- And they can choose any bundle which satisfies their budget constraint

$$\left\{ x \in \mathbb{R}^n_+ \middle| \sum_{i=1}^{n} p_i x_i \leq I \right\}$$



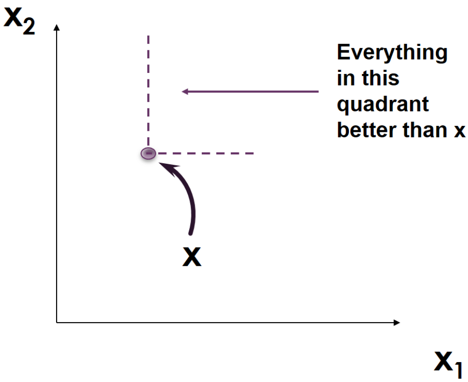
x_2
Budget constraint is
$p_1x_1 + p_2x_2 = I.$
x_1

Monotonicity

- Claim: We can use choice from budget sets to identify strict preference
  - Even if we only see a single bundle chosen from each budget set
- **As long as** we assume something about how preferences work
- One example: More is better

$$x_n \geq y_n \text{ for all } n \text{ and } x_n > y_n \text{ for some } n$$
$$\text{implies that } x \succ y$$

  - i.e. preferences are **strictly monotonic**

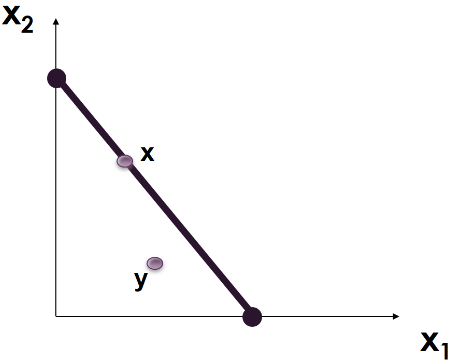

x2
Everything
in this
quadrant
better than x
x
x1

Monotonicity

- Claim: if p^x is the prices at which the bundle x was chosen

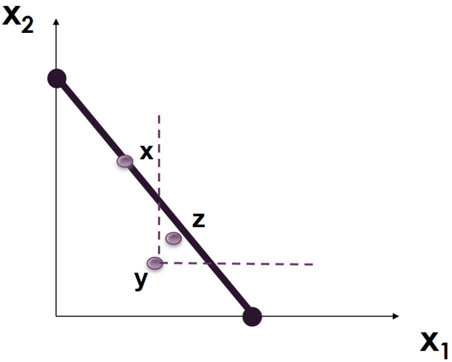
$$p^x x > p^x y \text{ implies } x \succ y$$

  - Why?



- Because x was chosen, we know $x \succsim y$
- Because $p^x x > p^x y$ we know that y was **inside** the budget set when x was chosen
- Could it be that $y \succsim x$?

Revealed Strictly Preferred



- Because y is inside the budget set, there is a z which is better than y **and** affordable when x was chosen
- Implies that $x \succsim z$ and (by monotonicity) $z \succ y$
- By transitivity $x \succ y$

- In fact we can make use of a weaker property than strict monotonicity

Definition

We say preferences $\succsim$ are **locally non-satiated** on a metric space X if, for every $x \in X$ and $\varepsilon > 0$, there exists

$$y \in B(x, \varepsilon)$$

such that

$$y \succ x$$

Lemma

Let x^j and x^k be two commodity bundles such that $p^j x^k < p^j x^j$. If the DM's choices can be rationalized by a complete locally non-satiated preference relation, then it must be the case that $x^j \succ x^k$

- When dealing with choice from budget sets we say
  - x is **directly revealed preferred to** y if $p^x x \geq p^x y$
  - x is **revealed preferred to** y if we can find a set of alternatives w_1, $w_2, ...w_n$ such that
    - x is directly revealed preferred to w_1
    - w_1 is directly revealed preferred to w_2
    - ...
    - w_{n-1} is directly revealed preferred to w_n
    - w_n is directly revealed preferred to y
  - x is **strictly revealed preferred to** y if $p^x x > p^x y$

Theorem (Afriat)

Let $\{x^1, ...x^I\}$ be a set of chosen commodity bundles at prices $\left\{p^1, ..., p^I\right\}$. The following statements are equivalent:

1. *The data set can be rationalized by a locally non-satiated set of preferences $\succeq$ that can be represented by a utility function*
2. *The data set satisfies GARP (i.e. xRy implies not ySx)*
3. *There exists positive $\left\{u^i, \lambda^i\right\}_{i=1}^{I}$ such that*

$$u^i \leq u^j + \lambda^j p^j (x^i - x^j) \; \forall \; i, j$$

4. *There exists a continuous, concave, piecewise linear, strictly monotonic utility function u that rationalizes the data*

Things to note about Afriat's Theorem

- Compare statement 1 and statement 4
  - The data set can be rationalized by a locally non-satiated set of preferences $\succeq$ that can be represented by a utility function
  - There exists a continuous, concave, piecewise linear, strictly monotonic utility function u that rationalizes the data
- This tells us that there is no empirical content to the assumptions that utility is
  - Continuous
  - Concave
  - Piecewise linear
- If a data set can be rationalized by any locally non-satiated set of preferences it can be rationalized by a utility function which has these properties

Things to note about Afriat's Theorem

- What about statement 3?
  - There exists positive $\left\{u^i, \lambda^i\right\}_{i=1}^{I}$ such that

$$u^i \leq u^j + \lambda^j p^j (x^i - x^j) \ \forall \ i, j$$

- This says that the data is rationalizable if a certain linear programming problem has a solution
  - Easy to check computationally
  - Less insight than GARP
  - But there are some models which do not have an equivalent of GARP but do have an equivalent of these conditions

Things to note about Afriat's Theorem

- Where do these conditions come from?
- Imagine that we knew that this problem was differentiable

$$\max u(x) \text{ subject to } \sum_j p^i_j x_j \leq I$$

  with u concave
- FOC for every problem i and good j

$$\frac{\partial u(x^i)}{\partial x^i_j} = \lambda^i p^i_j$$

- Implies

$$\nabla u(x^i) = \lambda^i p^i$$

  - where ∇u is the gradient function and p^i is the vector of prices

Things to note about Afriat's Theorem

- Recall (or learn), that for concave functions

$$u(x^i) \leq u(x^j) + \nabla u(x^j)(x^i - x^i)$$

  - i.e. function lies below the tangent
- So

$$u(x^i) \leq u(x^j) + \lambda^j p^j (x^i - x^j)$$

What if X is not Finite?

- So far we have assumed that the set of available alternatives is finite

Theorem

A Choice Correspondence on a **finite** *X has a utility representation if and only if it satisfies axioms α and β*

- However, this may be limiting
  - Choice from lotteries
  - Choice from budget sets
- Can we drop the word 'finite' from the above theorem?

What if X is not Finite?

- Remember we proved the theorem in three steps
  1. Show that if the data satisfies α and β then we can find a complete, transitive, reflexive preference relation $\succeq$ which represents the data
  2. Show that if the preferences are complete, transitive and reflexive then we can find a utility function u which represents them
  3. Show that if the data has a utility representation then it must satisfy α and β
- If you go back and look carefully step 1 never made use of the fact that X was finite
- However, in step 2 we did
  - Proof by induction is only guaranteed to hold finitely

What if X is not Finite?

- Just because we made use of the fact that X was finite in that particular proof doesn't mean that it is necessary for the statement to be true
- Maybe we will be lucky and the statement remains true for arbitrary X....
- Sadly not

- Some definitions you should know

Definition

The natural, or counting numbers, denoted by $\mathbb{N}$, are the set of numbers $\{1, 2, 3, ...\}$

Definition

The integers, denoted by $\mathbb{Z}$, are the set of numbers $\{..., -3, -2, -1, 0, 1, 2, 3, ..\}$

Definition

The rational numbers, denoted by $\mathbb{Q}$, are the set of numbers

$$\mathbb{Q} = \left\{ \frac{a}{b} | a \in \mathbb{Z},\ b \in \mathbb{N} \right\}$$

Definition

A set is *countably infinite* if there is a bijection between that set and the natural numbers

- Here are some properties of $\mathbb{Q}$ and $\mathbb{R}$.

1. $\mathbb{Q}$ is countable
2. $\mathbb{R}$ is uncountable
3. For every $a, b \in \mathbb{R}$ such that $a < b$, there exists a $c \in \mathbb{Q}$ such $a < c < b$ (i.e. $\mathbb{Q}$ is dense in $\mathbb{R}$)

Definition

Let $\succeq$ be a binary relation on $\mathbb{R} \times \{1, 2\}$ such that

$$\begin{aligned} \{a, b\} &\succeq \{c, d\} \text{ iff} \\ \text{(i) } a &> c \\ \text{or (ii) } a &= c \text{ and } b \geq d \end{aligned}$$

You should check that you agree that $\succeq$ is a complete preference relation.

Fact

There is no utility function that rationalizes $\succeq$.

- **Proof:**
  - Assume that such a utility function exists
  - Then, for every $a \in \mathbb{R}$ it must be the case that $u(a, 2) > u(a, 1)$
  - Moreover, for every $b > a$

$$u(b, 1) > u(a, 2)$$

  - Thus, every $a \in \mathbb{R}$ generates an interval on the real line, and these intervals are non-overlapping
  - Each such interval includes a rational number
  - Contradicts the remark that the rational numbers are countable and the real numbers are not.

Utility Representation with Non-Finite X

- So what can we do in order to ensure that preferences have a utility representation?
- First things first: how big is the problem?
- The counter example above made use of the fact that X was uncountablein
- Does this mean the problem goes away if X is **countably** finite?
- It turns out the answer is yes

Theorem

If a relation $\succeq$ on a **countable** *X is complete, transitive and reflexive then there exists a utility function $u : X \to \mathbb{R}$ which represents $\succeq$, i.e.*

$$u(x) \geq u(y) \iff x \succeq y$$

Utility Representation with Countable X

- Proof:
  - Let $\{x_n\}$ be an enumeration of X
  - Let $x_0 = 0$
  - Assign a utility number u to each x_{n+1} as in the finite case, by using the utility representation that worked for $x_1, ...x_n$ and then assigning a number that works for x_{n+1}
  - This procedure assigns utility numbers to each $x \in X$
  - And we know that for any x_n the utility function represents preferences between x_n and x_m for $m \leq n$
  - Now take $x, y \in X$. WLOG $x = x_n$, $y = x_m$ for $m \leq n$
  - We know that $x \succeq y \iff x_n \succeq x_m \iff u(x_n) \geq u(x_m)$
- Why does this proof not work if X is uncountable?

Utility Representation with Uncountable X

- We know from the example of lexicographic preferences that we cannot replace 'countable' with 'any' X in the previous theorem
- In order to guarantee that we have a utility representation of a preference relation on an uncountable X we need another condition

- One way to go is to insist that preferences are **continuous**
- Broadly speaking, this means that if we change the items a a little bit the preferences also change only a little bit
- i.e. they don't 'jump'

Definition

We say that a preference relation $\succeq$ on a metric space X is continuous if, for any $x, y \in X$ such that $x \succ y$, there exists an $\varepsilon > 0$ such that, for any $x' \in B(x, \varepsilon)$ and $y' \in B(y, \varepsilon)$, $x' \succ y'$

- Examples of preferences that are not continuous?
  - I like to drink a bottle of wine in the evenings. If I cannot drink a full bottle then I would prefer not to drink
  - Lexicographic preferences (see homework)

- An alternative characterization of continuity:

Lemma

A preference relation $\succeq$ on a metric space X is continuous if and only if the set $\{(x, y) | x \succeq y\} \subset X \times X$ is closed

- i.e. For any $\{x_n, y_n\}_{n=1}^{\infty}$ such that $x_n \succeq y_n$ and $\lim_n \{x_n, y_n\} = \{x, y\}$ implies $x \succeq y$
- You will prove for homework that these two definitions are equivalent

- One of the most famous theorems in mathematical social sciences

Theorem (Debreu)

Let X be a separable metric space, and $\succeq$ be a complete preference relation on X. If $\succeq$ is continuous, then it can be represented by a continuous utility function.

- Proving this in all its glory is beyond us, so we are going to prove something weaker

Theorem

Let X be a convex subset of $\mathbb{R}^n$ and $\succeq$ be a complete preference relation on X. If $\succeq$ is continuous, then it can be represented by a utility function.

Lemma

If $\succeq$ is a continuous complete preference relation on a convex subset of $\mathbb{R}^n$ and $x \succ y$ then there exists $z \in X$ such that $x \succ z \succ y$

- **Proof:** Assume not
  - Construct the following sequence inductively
  - Set $x_0 = x$ and $y_0 = 0$
  - At step $n+1$ assume that $x_n \succeq x$ and $y \succeq y_n$
  - Take the point m between x_n and y_n
  - It must be the case that either $m \succeq x$ or $y \succeq m$ (otherwise we have $x \succ m \succ y$ which we have ruled out by assumption)
  - In the former case set x_{n+1} to m and y_{n+1} to y_n. In the latter case, set x_{n+1} to x_n and y_{n+1} to m
  - This generates two sequences which converge to the same point z
  - By continuity of preferences, as $x_n \succeq x$ for every n it must be $z \succeq x$
  - Similarly, as $y \succeq y_n$ every n it must be that $y \succeq z$
  - Implies by transitivity that $y \succeq x$ - contradiction

- We will need one more definition

Definition

A set Y is **dense** in the set X if, for every $x \in X$ and $\varepsilon > 0$ there exists $y \in Y$ in $B(x, \varepsilon)$

Fact

$\mathbb{R}^n$ has a countable dense subset (e.g. the members of $\mathbb{R}^n$ where each coordinate is rational)

- We can now prove our theorem
- **Step 1:** Let Y be a countable dense subset of X. We have already shown that there exists a function v which represents $\succeq$ on Y.
  - In fact, we can restrict this function to be between -1 and 1
- **Step 2:** Define u as follows. For any $x \in X$

$$u(x) = \sup \{v(z) | z \in Y \text{ and } x \succ z\}$$

  - If no y exists such that $x \succ y$ let $u(x) = -1$

- **Step 3:** We now need to show that u represents $\succeq$. We can do that in two parts
  - First note that if $x \sim y$ then $x \succ z$ if and only if $y \succ z$ and so

$$\begin{aligned} u(x) &= \sup\{v(z)|z \in Y \text{ and } x \succ z\} \\ &= \sup\{v(z)|z \in Y \text{ and } y \succ z\} \\ &= u(y) \end{aligned}$$

- **Step 4:** If $x \succ y$ then, by previous lemma, there exists z_1 and z_2 such that $x \succ z_1 \succ z_2 \succ y$
  - By continuity this means that we can pick z_3 and $z_4 \in Y$ such that $x \succ z_3 \succ z_4 \succ y$
  - Thus

$$\begin{aligned} u(x) &\geq u(z_3) \\ &> u(z_4) \\ &\geq u(y) \end{aligned}$$