# Consumer Choice 1 - Proofs

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- Another property often assumed is **convexity**
  - The preference relation $\succeq$ is convex if the upper contour set $U_{\succeq}(x) = \{y \in X | y \succeq x\}$ is convex
  - i.e. for any $x, z, y$ such that $y \succeq x$ and $z \succeq x$ and $\alpha \in (0,1)$

$$(\alpha y + (1-\alpha)z) \succeq x$$

- What do convex indifference curves look like?
- Some alternative (equivalent) definitions of convexity
  - If $x \succeq y$ then for any $\alpha$ $(0,1)$ $\alpha x + (1-\alpha)y \succeq y$
  - For all $z = \alpha x + (1-\alpha)y$ either $z \succeq x$ or $z \succeq y$

- Proof that all are equivalent
  - $1 \Rightarrow 2$: Direct: $x \succeq y$ and $y \succeq y$ so $\alpha x + (1-\alpha)y \succeq y$
  - $2 \Rightarrow 3$: Either $x \succeq y$ in which case $z \succeq y$ or $y \succeq x$ and so $z \succeq x$
  - $3 \Rightarrow 1$: WLOG assume $y \succeq z$, then $\alpha y + (1-\alpha)z \succeq z \succeq x$

### Fact

*A complete preference relation with a utility representation is convex if and only if it can be represented by a quasi concave utility function - i.e., for every x the set*

$$\{y \in X | u(y) \geq u(x)\}$$

*is convex*

### Proof.

Immediate let $y \succeq x$ and $z \succeq x$ then $u(y) \geq u(x)$ and $u(z) \geq u(x)$, by q concavity $u(\alpha y + (1-\alpha)z) \geq u(x)$ and so $\alpha y + (1-\alpha)z \succeq x$
Similarly, say $u(y) \geq u(x)$ and $u(z) \geq u(x)$ implies $y \succeq x$ and $z \succeq x$ by convexity $\alpha y + (1-\alpha)z \succeq x$ and so $u(\alpha y + (1-\alpha)z) \geq u(x)$ □

## Fact

*A complete, strictly monotonic, continuous homothetic preference relation with a utility representation can be represented with a utility function which is homogenous of degree 1, i.e.*

$$u(\alpha x_1, ...\alpha x_n) = \alpha u(x_1, ...x_n)$$

## Proof.

*You have shown for homework that a utility representation for such preferences is given by*

$$u(x) = a | \{a, a, ...a\} \sim x$$

*If* $\succeq$ *is homothetic, then*

$$\begin{aligned} \{a, a, ...a\} &\sim x \rightarrow \{\alpha a, \alpha a, ...\alpha a\} \sim \alpha x \\ &\Rightarrow u(\alpha x) = \alpha a = \alpha u(x) \end{aligned}$$

### Fact

*A complete, continuous strictly monotonic, quasi linear preference relation with a utility representation can be represented with a utility function of the form*

$$u(x) = v(x_2, ...x_k) + x_1$$

### Lemma

*Under the conditions of the proof, for any* $x_2, ....x_k$ *there exists a* $v(x_2, ....x_k)$ *such that*

$$(0, x_2, ...x_k) \sim (v(x_2, ....x_k), 0, ..., 0)$$

### Lemma

### Proof.

*(for k=2): We want to want to show that for every $x_2$ there is some $x_1$ such that $(x_1, 0) \sim (0, x_2)$. Thus we need to show that the following set is empty*

$$T = \{t | (0, t) \succ (x_1, 0) \ \forall \ x_1\}$$

*Assume not* □

Proof.

[Proof (cont)] First assume $m \in T$. Then $m > 0$ and $(1, m) \succ (0, m)$ (MON) and an $\varepsilon$ such that $(1, m - \varepsilon) \succ (0, m)$ (CONT) and so $(1, m - \varepsilon) \succ (x_1 + 1, 0)\ \forall\ x_1$ (as $M \in T$). As $m = \inf T$, there exists an $x_1^*$ such that $(x_1^*, 0) \succeq (0, m - \varepsilon)$, but by quasi linearity $(x_1^* + 1, 0) \succeq (1, m - \varepsilon)$ a contradiction
Next assume that $m \notin T$. Then $(x_1^*, 0) \sim (0, m)$ for some $x_1^*$, and by monotonicity $(x_1^* + 1, 0) \succ (0, m)$. By continuity there exists an $\varepsilon > 0$ such that $(x_1^* + 1, 0) \succ (0, x_2)$ for any $m + \varepsilon > x_2 > m$ meaning that such $x_2$ are not in $T$ and so $m \neq \inf T$
Thus, by the standard argument, we can set $v(x_2) = \inf(x_1 | (x_1, 0) \succeq (0, x_2))$. We have just show that this set is non-empty, and by continuity we know that this implies that $(v(x_2), 0) \sim (0, x_2)$ □

Proof.

[Proof (of Claim)] We want to show that $x_1 + v(x_2, ...x_k)$ represents $\succeq$. Note that for any $x$

$$\begin{aligned}(0, x_1, ....x_k) &\sim (v(x_2, ..., x_k), 0, ...0) \Rightarrow \\ (x_1 + v(x_2, ..., x_k), 0...0) &\sim x \text{ (by q linearity)}\end{aligned}$$

Thus by strong monotonicity we are done

$$\begin{aligned}x &\succeq y \\ &\Rightarrow (x_1 + v(x_2, ..., x_k), 0...0) \succeq (y_1 + v(y_2, ..., y_k), 0...0)\end{aligned}$$

□

**Theorem**

*If $\succeq$ are strongly monotonic, convex, continuous and differentiable*$^*$*, and*

1. $px^* = w$
2. *for every $k$ such that $x_k^* > 0$*

$$\frac{\frac{\partial u(x^*)}{\partial x_k}}{p_k} \geq \frac{\frac{\partial u(x^*)}{\partial x_j}}{p_j}$$

*Then $x^*$ is a solution to the consumer's problem*

Proof.

Assume not, then there exists a $y$ such that $py \leq px^*$ but $y \succ x^*$

Let $\mu = \frac{\frac{\partial u(x^*)}{\partial x_k}}{p_k}$ for all $k$ st $x_k^* > 0$ and note

$$\begin{aligned} 0 &\geq p(y - x^*) \\ &= \sum_k p^k (y_k - x_k^*) \\ &\geq \sum_k \frac{\partial u(x^*)}{\partial x_k} \frac{(y_k - x_k^*)}{\mu} \end{aligned}$$

This follows from the fact that if $x_k^* > 0$ then $p^k = \frac{\partial u(x^*)}{\partial x_k} / \mu$, and if not $(y_k - x_k^*) \geq 0$ and $p^k \geq \frac{\partial u(x^*)}{\partial x_k} / \mu$

Thus

$$0 \geq \nabla u(x^*)(y - x^*)$$

□

Proof.

But the fact that $y \succ x^*$, along with strong monotonicity, continuity, and convexity means that

$$\lambda x^* + (1-\lambda)y \succ x^*$$

(see Rubinstein)

Thus $(y - x^*)$ is an improving direction - a contradiction □

**Theorem**

*Let $x$ be a rationalizable demand function that satisfies Walras' law and $I' = p'x(p, I)$. Then*

$$[p' - p][x(p', I') - x(p, I)] \leq 0$$

**Proof.**

*Assume $p$*

$$\begin{aligned} & [p' - p][x(p', I') - x(p, I)] \\ = & \; p'x(p', I') - p'x(p, I) - px(p', I') + px(p, I) \\ = & \; I' - I' + px(p, I) - px(p', I') \end{aligned}$$

*If $px(p.I) - px(p', I') > 0$ then $x(p.I)$ strictly preferred to $x(p', I')$*
*But as $p'x(p.I) = I'$ $x(p', I')$ weakly preferred to $x(p.I)$* □

# Properties of the Indirect Utility Function

- **Property 3:** $v$ is quasiconvex: i.e. the set

$$\{(p,w)|v(p,w) \leq \bar{v}\}$$

is convex for all $\bar{v}$

Proof.

Take $p, w$ and $p', w'$ in the set and let $p'' = \alpha p + (1-\alpha)p'$ and $w'' = \alpha w + (1-\alpha)w'$.

NTS that $v(p'', w'') \leq \bar{v}$. Assume not. Then there exists an $x$ st $u(x) > \bar{v}$ and

$$\begin{aligned} p''x &\leq w'' \Rightarrow \\ \alpha px + (1-\alpha)p'x &\leq \alpha w + (1-\alpha)w' \Rightarrow \\ \alpha(px - w) + (1-\alpha)(p'x - w') &\leq 0 \end{aligned}$$

□

Proof.

Contradiction □

**Theorem**

*If $u$ is monotonic and continuous then if $x^*$ is a solution to the prime problem with prices $p$ and wealth $w$ it is a solution to the dual problem with prices $p$ and utility $v(p, w)$*

Proof.

- Assume not, then there exists a bundle $\bar{x}$ such that

$$u(\bar{x}) \geq v(p, w) = u(x^*)$$

with

$$\sum p_i \bar{x}_i < \sum p_i x_i^* = w$$

- But this means, by monotonicity, that there exists an $\varepsilon > 0$ such that, for

$$x' = \begin{pmatrix} \bar{x}_1 + \varepsilon \\ \bar{x}_2 + \varepsilon \\ \vdots \\ \bar{x}_N + \varepsilon \end{pmatrix}$$

$$\sum p_i x_i' < w$$

□

Proof.

- By monotonicity, we know that $u(x') > u(\bar{x}) \geq u(x^*)$, and so $x^*$ is not a solution to the prime problem

□

**Theorem**

*If $u$ is monotonic and continuous then if $x^*$ is a solution to the dual problem with prices $p$ and utility $u^*$ it is a solution to the prime problem with prices $p$ and wealth $\sum p_i x_i^*$*

Proof.

- Assume not, then there exists a bundle $\bar{x}$ such that

$$\sum p_i \bar{x}_i \leq \sum p_i x_i^*$$

with

$$u(\bar{x}) > u(x^*) \geq u^*$$

- By continuity, there exists an $\varepsilon > 0$ such that, for all $x' \in B(\bar{x}, \varepsilon)$, $u(x') > u(x^*)$

□

Proof.

- In particular, there is an $\varepsilon > 0$ such that

$$x' = \begin{pmatrix} \bar{x}_1 - \varepsilon \\ \bar{x}_2 - \varepsilon \\ \vdots \\ \bar{x}_N - \varepsilon \end{pmatrix}$$

  and $u(x') > u(x^*) \geq u^*$

- But $\sum p_i x_i' < \sum p_i \bar{x}_i \leq \sum p_i x_i^*$, so $x^*$ is not a solution to the dual problem.

□

# Properties of the Hicksian Demand Function

- **Fact 3**: If preferences are convex then $h$ is a convex set. If preferences are strictly convex then $h$ is unique
  - Proof: say that $x$ and $y$ are both in $h(p, u)$. Then
  $$\sum p_i x_i = \sum p_i y_i = e(p, u)$$
  - Implies that for any $\alpha \in (0, 1)$ and $z = \alpha x + (1 - \alpha) y$
  $$\begin{aligned} \sum p_i z_i &= \sum p_i \left(\alpha x_i + (1 - \alpha) y_i\right) \\ &= \alpha \sum p_i x_i + (1 - \alpha) \sum p_i y_i \\ &= e(p, u) \end{aligned}$$
  - Also, as preferences are convex, $z \succeq x$, and so $u(z) \geq u(x) = u$
  - If preferences are **strictly** convex, then $z \succ x$
  - But, by continuity, exists $\varepsilon > 0$ such that $z' \succ x$ all $z' \in B(z, \varepsilon)$
  - Implies that there is a $z'$ such that $u(z') > u$ and $\sum p_i z_i' < \sum p_i x_i$

# Properties of the Expenditure Function

- **Fact 4:** $e$ is concave in $p$
  - Proof: fix a $\bar{u}$. we need to show that

$$e(p'', \bar{u}) \geq \alpha e(p, \bar{u}) + (1-\alpha) e(p', \bar{u})$$

where

$$p'' = \alpha p + (1-\alpha) p'$$

  - Let $x'' \in h(p'', \bar{u})$, then

$$\begin{aligned} e(p'', \bar{u}) &= \sum p_i'' x_i'' \\ &= \sum \left( \alpha p_i + (1-\alpha) p_i' \right) x_i'' \\ &= \alpha \sum p_i x_i'' + (1-\alpha) \sum p_i' x_i'' \\ &\geq \alpha \sum p_i x_i + (1-\alpha) \sum p_i' x_i' \\ &= \alpha e(p, \bar{u}) + (1-\alpha) e'(p, \bar{u}) \end{aligned}$$

where $x \in h(p, \bar{u})$ and $x' \in h(p', \bar{u})$