Microeconomic Analysis

Mark Dean

Homework 5

Due Thursday October 10th

Question 1 Here are five commonly used utility function for the case of two commodities

- Cobb-Douglas: $u(x_1, x_2) = x_1^\alpha x_2^\beta$ for $\alpha > 0$ and $\beta > 0$
- Constant Elasticity of Substitution: $u(x_1, x_2) = (\alpha_1 x_1^\rho + \alpha_2 x_2^\rho)^{\frac{1}{\rho}}$ for $\rho \leq 1$ and $\alpha_1, \alpha_2 > 0$
- Linear: $u(x_1, x_2) = \alpha x_1 + \beta x_2$ for $\alpha > 0$ and $\beta > 0$
- Leontief: $u(x_1, x_2) = \min(\alpha x_1, \beta x_2)$ for $\alpha > 0$ and $\beta > 0$
- Quasi-linear: $u(x_1, x_2) = x_1 + x_2^\alpha$ for $\alpha > 0$

1. For each of these utility functions derive the Walrasian and Hicksian demand correspondences, the indirect utility function and the Slutsky matrix (where it is defined)
2. Show that the Walrasian and Hicksian demand functions of the CES preferences converge to those of the Linear and Leontief preferences as $\rho \to 1$ and $\rho \to -\infty$ respectively. For simplicity, you can focus only on cases in which all three preferences give demand functions which are single valued
3. The elasticity of substitution between two goods is given by

$$\xi_{1,2} = \frac{\partial(x_1(p, w)/x_2(p, w))}{\partial(p_1/p_2)} \frac{p_1/p_2}{x_1(p, w)/x_2(p, w)}$$

Provide an interpretation of the elasticity, and calculate it for each of the 5 preferences above

Question 2 Some questions on expected utility theory. Assume throughout that preferences are transitive and complete

1. Show that choices made by an expected utility maximizer must satisfy the Independence axiom and the 'Archemedian Axiom' form of continuity - i.e. that for p, q and r such that $p \succ q \succ r$, there must exist an a and b in $(0, 1)$ such that

$$ap + (1 - a)r \succ q \succ bp + (1 - b)r$$

2. Prove that, if a function $u : X \to \mathbb{R}$ forms an expected utility representation for some preferences $\succeq$ then $v : X \to \mathbb{R}$ will also do so if and only if there exists an $a > 0$ and b such that

$$u(x) = av(x) + b \; \forall \; x \in X$$

3. Recall that our standard definition of continuity stated that if $p \succ q$ then there exists an $\varepsilon > 0$ such that $p' \in B(p, \varepsilon)$ and $q' \in B(q, \varepsilon)$ implies that $p' \succ q'$ where $B(., \varepsilon)$ is the $\varepsilon-$ball. What is the relationship between this and the Archemedian Axiom form of Continuity - i.e. does either one imply the other? (if you worry about this type of thing, you can metricize the space of lotteries using the Euclidean distance between the probability vectors).

4. Consider the following two decision making procedures. Do they satisfy Independence and 'Archemedian Axiom' continuity?

   (a) *Lexicographic preferences*: The prizes are ordered $z_1, ..., z_k$. When comparing lotteries p and q the DM first compares the probability of prize z_1. If one lottery offers a higher probability they prefer that lottery. If $p(z_1) = q(z_1)$ they compare prize z_2 and so on.

   (b) *Compare the median prize:* The decision maker assigns a utility number to each prize and uses those numbers to order the prizes. Label the prizes $x_1, .., x_N$ so that $u(x_1) \leq u(x_2) \leq, ... \leq u(x_N)$. The median prize is the prize x_n such that

$$\sum_{k=1}^{n} p(x_k) > 0.5 \geq \sum_{k=1}^{n-1} p(x_k)$$

   The decision maker assigns to the lottery the utility of the median prize.

Question 3 This is a question which explores the type of preferences we used to construct the indirect utility function - sometimes called preferences over menus.

Consider the following decision maker: They have a utility function u on a set of finite alternatives X. Their preferences over sets (or menus) of these alternatives (which we indicate by $\trianglerighteq$ for weak preferences) are given by the following. For any $A \in 2^X/\emptyset$, $B \in 2^X/\emptyset$

$$\begin{aligned} A &\trianglerighteq B \text{ if and only if} \\ \max_{x\in A} u(x) &\geq \max_{x\in B} u(x) \end{aligned}$$

1. Show that the binary relation $\trianglerighteq$ is a preference relation
2. Show that it satisfies the following property: if $A \trianglerighteq B$, then $A \bowtie A \cup B$ (where $\bowtie$ indicates indifference - i.e. $A \bowtie B$ iff $A \trianglerighteq B$ and $B \trianglerighteq A$)
3. Show that if $\trianglerighteq$ is a preference relation that has this property then we can find a utility function $u : X \rightarrow \mathbb{R}$ such that $A \trianglerighteq B$ if and only if $\max_{x\in A} u(x) \geq \max_{x\in B} u(x)$ (i.e. we can find a representation of the type in part 1). *Hint: in the data we have, what observation tells us that an alternative $x \in X$ is preferred to an alternative $y \in X$?*
4. Consider a modification of the above model so that utility depends on some state $s \in S$ (with S finite) which is not known when the menu is chosen, but will be known when the choice is made from that menu. The value of the menu is the expected value of the resulting choice, so

$$\begin{aligned} A &\trianglerighteq B \text{ if and only if} \\ \sum_{s\in S} \max_{x\in A} u(x,s)p(s) &\geq \sum_{s\in S} \max_{x\in B} u(x,s)p(s) \end{aligned}$$

   where $p(.)$ is the probability distribution over S at the time when the menu is chosen.

   (a) Show that the property from part (2) no longer holds

   (b) Under this model, if $A \trianglerighteq B$, what can we say about the relationship between A and $A \cup B$