Risk and Uncertainty - Proofs

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GR6211 - Microeconomic Analysis 1

Lemma If $\succeq$ is a preference relation that satisfies Independence then $p \succ q$ and $0 \leq \beta < \alpha \leq 1$ implies

$$\alpha p + (1-\alpha)q \succ \beta p + (1-\beta)q$$

Proof.

By indepdendence

$$\alpha p + (1-\alpha)q \succ \alpha q + (1-\alpha)q = q$$

Applying independence again gives

$$\begin{aligned} & \alpha p + (1-\alpha)q \\ = \ & (1-\frac{\beta}{\alpha})\left(\alpha p + (1-\alpha)q\right) + \frac{\beta}{\alpha}\left(\alpha p + (1-\alpha)q\right) \\ \succ \ & (1-\frac{\beta}{\alpha})q + \frac{\beta}{\alpha}\left(\alpha p + (1-\alpha)q\right) \\ = \ & \beta p + (1-\beta)q \end{aligned}$$

Proof

Lemma If $\succeq$ is a preference relation that satisfies Independence and Continuity then $p \succeq q \succeq r$ and $p \succ r$ implies that there exists a unique α^* such that

$$q \sim \alpha^* p + (1 - \alpha^*) r$$

Proof.

Trivial if $p \sim q$ or $r \sim q$ so assume not. Let

$$\alpha = \inf \{\alpha | \alpha p + (1 - \alpha) r \succ q\}$$

Note that by Continuity and the previous lemma $\hat{\alpha} \in (0, 1)$

NTS that $q \sim \hat{\alpha} p + (1 - \hat{\alpha}) r$ □

Proof

- Say
$$p \succ q \succ \hat{\alpha} p + (1 - \hat{\alpha}) r \succ r$$
Then by continuity there exists β such that
$$q \succ \beta p + (1 - \beta) \left(\hat{\alpha} p + (1 - \hat{\alpha}) r \right)$$
By monotonicity, $\beta + (1 - \beta)\hat{\alpha} > \hat{\alpha}$ is a lower bound, so $\hat{\alpha}$ is not the greatest lower bound
- Say
$$p \succ \hat{\alpha} p + (1 - \hat{\alpha}) r \succ q \succ r$$
Then by continuity there exists β such that
$$\beta \left(\hat{\alpha} p + (1 - \hat{\alpha}) r \right) + (1 - \beta) r \succ q$$
So, as $\beta\hat{\alpha} < \hat{\alpha}$ cannot be a lower bound

Proof

- Back to main proof
- Define $\trianglerighteq$ on X as

$$x \trianglerighteq y \text{ if } \delta_x \succeq \delta_y$$

- Note that $\trianglerighteq$ is a preference relation (check!)
- Pick x^* which is $\trianglerighteq$ maximal and x_* which is $\trianglerighteq$ minimal
- Note it must be the case that

$$x^* \succeq p \succeq x_* \text{ all } p$$

- Note that if $\delta_x \sim \delta_y \ \forall \ x, y \in X$ then proof is trivial (set all utilities to zero)

- Assign utilities in the following way
  1. Let $u(x^*) = 1$
  2. Let $u(x_*) = 0$
  3. For all other $x \in X$ let

$$u(x) = \alpha \text{ st } x \sim \alpha x^* + (1-\alpha)x_*$$

- So now we have found utility numbers for every prize
- All we have to do is show that $p \succeq q$ if and only if $\sum_{x \in X} p_x u(x) \geq \sum_{x \in X} q_x u(x)$
- Let's do a simple example for a 4 prize case with $p = \{p(a), p(b), p(c), p(d)\}$
  - assume $a = x^*$ and $d = x_*$

$$p = \begin{pmatrix} 0 \\ 0.25 \\ 0.75 \\ 0 \end{pmatrix}, \quad q = \begin{pmatrix} 0 \\ 0.75 \\ 0.25 \\ 0 \end{pmatrix}$$

- First, notice that

$$p = \begin{pmatrix} 0 \\ 0.25 \\ 0.75 \\ 0 \end{pmatrix} = 0.25 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 0.75 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

- But

- But

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \sim u(b) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + (1 - u(b)) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \sim u(c) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + (1 - u(c)) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$
\begin{aligned}
p \quad \sim \quad & 0.25 \left(u(b) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + (1 - u(b)) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right) \\
& +0.75 \left(u(c) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + (1 - u(c)) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right)
\end{aligned}
$$

$$= \quad (0.25u(b) + 0.75u(c)) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} +$$

$$(1 - 0.25u(b) - 0.75u(c)) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

- So p is indifferent to a lottery that puts probability

$$(0.25u(b) + 0.75u(c))$$

  on the best prize (and the remainder on the worst prize)
- **But this is just the expected utility of** p
- Similarly q is indfferent to a lottery that puts

$$(0.75u(b) + 0.25u(c))$$

  on the best prize
- **But this is just the expected utility of** q

The Expected Utility Theorem

- So p will be preferred to q if the expected utility of p is higher than the expected utility of q
- This is because this means that p is indifferent to a lottery which puts a higher weight on the best prize than does q
- QED (ish)