A Representation Theorem for Utility Maximization: Proofs

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- Our job is to show that, if choices satisfy α and β then we can find a preference relation $\succeq$ which is
  - Complete, transitive and reflexive
  - Represents choices

Theorem

A Choice Correspondence can be represented by a complete, transitive, reflexive preference relation if satisfies axioms α and β

- How should we proceed?
  1. Choose a candidate binary relation $\trianglerighteq$
  2. Show that it is a preference relation - i.e. complete and tranistive
     - Note that completeness implies reflexivity, so we dont have to check that separately
  3. Show that it represents choice

Guessing the Preference Relation

- If we observed choices, what do we think might tell us that x is preferred to y?
- How about if x is chosen when the only option is y?
- Let's try that!
- We will **define** $\trianglerighteq$ as saying

$$x \trianglerighteq y \text{ if } x \in C(x, y)$$

- **Remember this translation!**
  - Whenever I ask "what does it mean that $x \trianglerighteq y$"
  - You reply "x was chosen from the set $\{x, y\}$"
- Okay, great, we have defined $\trianglerighteq$
- But we need it to have the right properties

Completeness

- Is $\trianglerighteq$ **complete?**
- Yes!
- For any set $\{x, y\}$ either x or y must be chosen (or both)
- In the former case $x \trianglerighteq y$
- In the latter $y \trianglerighteq x$

Transitivity

- Is $\trianglerighteq$ **transitive?**
- Yes! (though this requires a little proving)
- Assume not, then

$$\begin{aligned} x &\trianglerighteq y,\ y \trianglerighteq z \\ \text{but not } x &\trianglerighteq z \end{aligned}$$

- We need to show that this **cannot happen**
- i.e. it violates α or β
- These are conditions on the data, so what do we need to do?
- Understand what this means for the data

Transitivity

- Translating to the data
  - $x \trianglerighteq y$ means that $x \in C(x, y)$
  - $y \trianglerighteq z$ means that $y \in C(y, z)$
  - not $x \trianglerighteq z$ means that $x \notin C(x, z)$
- Claim: such data cannot be consistent with α and β
- Why not?

Transitivity

- What would the person choose from $\{x, y, z\}$
- x?
  - No! Violation of α as x not chosen from $\{x, z\}$
- y?
  - No! This would imply (by α) that $y \in C(x, y)$
  - By β this means that $x \in C(x, y, z)$
  - Already shown that this can't happen
- z?
  - No! This would imply (by α) that $z \in C(y, z)$
  - By β this means that $y \in C(x, y, z)$
  - Already shown that this can't happen

Transitivity

- If we have $x \trianglerighteq y$, $y \trianglerighteq z$ but not $x \trianglerighteq z$ then the data cannot satisfy α and β
- Thus if α and β are satisfied, we know that $\trianglerighteq$ must be transitive!
- Thus, we can conclude that, if α and β are satisfied $\trianglerighteq$ must have all three right properties!

- Finally, we need to show that $\trianglerighteq$ represents choices - i.e.

$$C(A) = \{x \in A | x \trianglerighteq y \text{ for all } y \in A\}$$

- How do we do this?
- Well, first note that we are trying to show that two **sets** are equal
    - The set of things that are chosen
    - The set of things that are best according to $\trianglerighteq$
- We do this by showing two things
    1. That if x is in $C(A)$ it must also be $x \trianglerighteq y$ for all $y \in A$
    2. That if $x \trianglerighteq y$ for all $y \in A$ then x is in $C(A)$

Things that are Chosen must be Preferred

- Say that $x \in C(A)$
- For $\trianglerighteq$ to represent choices it must be that $x \trianglerighteq y$ for every $y \in A$
- Note that, if $y \in A$, $\{x, y\} \subset A$
- So by α if

$$\begin{aligned} x &\in C(A) \\ &\Rightarrow x \in C(x, y) \end{aligned}$$

- And so, by definition

$$x \trianglerighteq y$$

Things that are Preferred must be Chosen

- Say that $x \in A$ and $x \trianglerighteq y$ for every $y \in A$
- Can it be that $x \notin C(A)$
- No! Take any $y \in C(A)$
- By α, $y \in C(x, y)$
- As $x \trianglerighteq y$ it must be the case that $x \in C(x, y)$
- So, by β, $x \in C(A)$
- Contradiction!

Done!

Q.E.D.

- Well, unfortunately we are not really done
- We wanted to test the model of **utility maximization**
- So far we have shown that α and β are equivalent to preference maximization
- Need to show that preference maximization is the same as utility maximization

Theorem

If $\succeq$ is a preference relation on a finite X then there exists a utility function $u : X \to \mathbb{R}$ which represents $\succeq$, i.e.

$$u(x) \geq u(y) \Longleftrightarrow x \succeq y$$

Proof By Induction

- We are going to proceed using **proof by induction**
  - We want to show that our statement is true regardless of the size of X
  - We do this using induction on the size of the set
  - Let $n = |X|$, the size of the set
- Induction works in two stages
  - Show that the statement is true if $n = 1$
  - Show that, if it is true for n, it must also be true for any $n + 1$
- This allows us to conclude that it is true for n
  - It is true for $n = 1$
  - If it is true for $n = 1$ it is true for $n = 2$
  - If it is true for $n = 2$, it is true for $n = 3$....
- You have to be a bit careful with proof by induction
  - Or you can prove that all the horses in the world are the same color

From Preference To Utility

- So in this case we have to show that we can find a utility representation if $|X| = 1$
  - Trivial
- And show that if a utility representation exists for $|X| = n$, then it exists for $|X| = n + 1$
  - Not trivial

Step 1

- Take a set such that $|X| = n + 1$ and a complete, transitive reflexive preference relation $\succeq$
- Remove some $x^* \in X$
- Note that the new set X/x^* has size n
  - And that the binary relation $\succeq$ restricted to this set is still a preference relation
- So, by the inductive assumption, there exists some $v : X/x^* \rightarrow \mathbb{R}$ such that

$$v(x) \geq v(y) \Longleftrightarrow x \succeq y$$

- So now all we need to do is assign a utility number to x^* which makes it work with v
- How would you do this?

Step 2

- Four possibilities
  1. $x^* \sim y$ for some $y \in X/x^*$
     - Set $v(x^*) = v(y)$
  2. $x^* \succ y$ for all $y \in X/x^*$
     - Set $v(x^*) = \max_{y \in X/x^*} v(y) + 1$
  3. $x^* \prec y$ for all $y \in X/x^*$
     - Set $v(x^*) = \min_{y \in X/x^*} v(y) - 1$
  4. None of the above

Step 2

- What do we do in case 4?
- We divide X in two: those objects better than x^* and those worse than x^*

$$X_* = \{y \in X / x^* | x^* \succeq x\}$$

$$X^* = \{y \in X / x^* | x \succeq x^*\}$$

- Figure out the highest utility in X_* and the lowest utility in X^* and fit the utility of x^* in between them

$$v(x^*) = \frac{1}{2} \min_{y \in X^*} v(y) + \frac{1}{2} \max_{y \in X_*} v(y)$$

Step 2

- Note that everything in X^* has higher utility than everything in X_*
  - Pick an $x \in X^*$ and $y \in X_*$
  - $x \succeq x^*$ and $x^* \succeq y$
  - Implies $x \succeq y$ (why?)
  - and so $v(x) \geq v(y)$
  - In fact, because we have ruled out indifference $v(x) > v(y)$
- This implies that

$$v(x) > v(x^*) > v(y)$$

- And so
  - The utility of everything better than x^* is higher than $v(x^*)$
  - The utility of everything worse than x^* is lower than $v(x^*)$

Step 3

- Verify that v represents $\succeq$ in all of the four cases
- That sounds exhausting
- I'll leave it for you to do for homework

Done!

Q.E.D.