Utility Maximization 2: Extensions - Proofs

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- **Proof:**
  - Assume that such a utility function exists
  - Then, for every $a \in \mathbb{R}$ it must be the case that $u(a, 2) > u(a, 1)$
  - Moreover, for every $b > a$

$$u(b, 1) > u(a, 2)$$

  - Thus, every $a \in \mathbb{R}$ generates an interval on the real line, and these intervals are non-overlapping
  - Each such interval includes a rational number
  - Contradicts the remark that the rational numbers are countable and the real numbers are not.

Utility Representation with Countable X

- Proof:
  - Let $\{x_n\}$ be an enumeration of X
  - Let $x_0 = 0$
  - Assign a utility number u to each x_{n+1} as in the finite case, by using the utility representation that worked for $x_1, ...x_n$ and then assigning a number that works for x_{n+1}
  - This procedure assigns utility numbers to each $x \in X$
  - And we know that for any x_n the utility function represents preferences between x_n and x_m for $m \leq n$
  - Now take $x, y \in X$. WLOG $x = x_n$, $y = x_m$ for $m \leq n$
  - We know that $x \succeq y \iff x_n \succeq x_m \iff u(x_n) \geq u(x_m)$
- Why does this proof not work if X is uncountable?

Theorem

If a preference relation $\succeq$ can be represented by a continuous utility function then it is continuous

Proof.

Assume $\succeq$ is not continuous, then there exists a sequence $x_n \longrightarrow x$ and $y_n \longrightarrow y$ such that

$$x_n \succeq y_n \text{ but } y \succ x$$

But this implies that $u(x_n) \geq u(y_n)\ \forall\ n$ but $u(y) > u(x)$ contradicting continuity of u.
To see this let $\delta = \frac{u(y)-u(x)}{2} > 0$ and note that by the continuity of u there must exist some ε such that for x_n and y_n such that

$$\begin{aligned} d(x_n, x) &\leq \varepsilon, |u(x_n) - u(x))| < \delta. \\ d(y_n, y) &\leq \varepsilon, |u(y_n) - u(y))| < \delta. \end{aligned}$$

Theorem

Proof.

As $x_n \to x$ and $y_n \to y$ this implies that there must be some N such that, for $n > N$

$$\begin{aligned} d(x_n, x) &\leq \varepsilon. \\ d(y_n, y) &\leq \varepsilon \end{aligned}$$

and so u

$$\begin{aligned} u(x_n) &< u(x) + \delta \\ u(y_n) &> u(y) - \delta. \end{aligned}$$

□

Theorem

Proof.
but

$$\delta + u(x) = \frac{u(y) + u(x)}{2} = u(y) - \delta.$$

So

$$u(y_n) > u(x_n)$$

A contradiction □

Lemma

If $\succeq$ is a continuous complete preference relation on a convex subset of $\mathbb{R}^n$ and $x \succ y$ then there exists $z \in X$ such that $x \succ z \succ y$

- **Proof:** Assume not
  - Construct the following sequence inductively
  - Set $x_0 = x$ and $y_0 = y$
  - At step $n+1$ assume that $x_n \succeq x$ and $y \succeq y_n$
  - Take the point m between x_n and y_n
  - It must be the case that either $m \succeq x$ or $y \succeq m$ (otherwise we have $x \succ m \succ y$ which we have ruled out by assumption)
  - In the former case set x_{n+1} to m and y_{n+1} to y_n. In the latter case, set x_{n+1} to x_n and y_{n+1} to m
  - This generates two sequences which converge to the same point z
  - By continuity of preferences, as $x_n \succeq x$ for every n it must be $z \succeq x$
  - Similarly, as $y \succeq y_n$ every n it must be that $y \succeq z$
  - Implies by transitivity that $y \succeq x$ - contradiction

- We will need one more definition

Definition

A set Y is **dense** in the set X if, for every $x \in X$ and $\varepsilon > 0$ there exists $y \in Y$ in $B(x, \varepsilon)$

Fact

$\mathbb{R}^n$ *has a countable dense subset (e.g. the members of* $\mathbb{R}^n$ *where each coordinate is rational)*

- We can now prove our theorem
- **Step 1:** Let Y be a countable dense subset of X. We have already shown that there exists a function v which represents $\succeq$ on Y.
  - In fact, we can restrict this function to be between -1 and 1
- **Step 2:** Define u as follows. For any $x \in X$

$$u(x) = \sup \{v(z) | z \in Y \text{ and } x \succ z\}$$

  - If no y exists such that $x \succ y$ let $u(x) = -1$

- **Step 3:** We now need to show that u represents $\succeq$. We can do that in two parts
    - First note that if $x \sim y$ then $x \succ z$ if and only if $y \succ z$ and so

$$
\begin{aligned}
u(x) &= \sup\{v(z)|z \in Y \text{ and } x \succ z\} \\
&= \sup\{v(z)|z \in Y \text{ and } y \succ z\} \\
&= u(y)
\end{aligned}
$$

- **Step 4:** If $x \succ y$ then, by previous lemma, there exists z_1 and z_2 such that $x \succ z_1 \succ z_2 \succ y$
    - By continuity this means that we can pick z_3 and $z_4 \in Y$ such that $x \succ z_3 \succ z_4 \succ y$
    - Thus

$$
\begin{aligned}
u(x) &\geq u(z_3) \\
&> u(z_4) \\
&\geq u(y)
\end{aligned}
$$

- **Proof: GARP implies representation**
- First, note that R is transitive (and without loss of generality we can assume it is reflexive)
- Also note that, by GARP, S is the asymmetric part of R
- This means that, by Sziplrajn's theorem there exists a complete preference relation $\succeq$ such that

$$xRy \text{ implies } x \succeq y$$
$$xSy \text{ implies } x \succ y$$

The Generalized Axiom of Revealed Preference

- All we need to show is that $\succeq$ represents choice, i.e

$$C(A) = \{x \in A | x \succeq y \text{ all } y \in A\}$$

- Again, need to show two things

1. $x \in C(A) \Rightarrow x \succeq y$ all $y \in A$
    - This follows from the fact that $x \in C(A) \Rightarrow xR^D y \ \forall \ y \in A$ and so $x \succeq y \ \forall \ y \in A$
2. $x \in A$ and $x \succeq y$ all $y \in A \Rightarrow x \in C(A)$
    - Assume by way of contradiction $x \notin C(A)$, and take $y \in C(A)$
    - This implies that ySx and so $y \succ x$ and therefore not $x \succeq y$
    - Contradiction