Decision Theory and Evidence

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Homework 1

Due Tuesday 24th Feburary

Question 1 Let $\succeq$ be a complete relation on a non-empty set X, and S a non-empty finite subset of X. Define

$$C_{\succeq}(S) = \{x \in S | x \succeq y \text{ for all } y \in S\}$$

1. Show that, if $\succeq$ is transitive, then $C_{\succeq}(S) \neq \emptyset$
2. Show that, if $\succeq$ is acyclic, then $C_{\succeq}(S) \neq \emptyset$
3. Show that, if $C_{\succeq}(S)$ is non-empty for every finite subset of X, then $\succeq$ must satisfy OWC.

   Now for for **any** binary relation $\succ$, show that the choice correspondence defined by $C(A) = \{x \in A | y \succ x \text{ for no } y \in A\}$ will satisfy property α **if this is indeed a choice correspondance.** Under what conditions on $\succ$, will $C(A)$ be a choice correspondence?

Question 2 Consider a preference relation $\succeq$ on some finite set X.

1. Which result that we showed in class means that there exists a utility function $u : X \to \mathbb{R}$ that represents $\succeq$ in the sense that

$$\begin{aligned} x &\succeq y \to u(x) \geq u(y) \\ x &\succ y \to u(x) > u(y) \end{aligned}$$

   (This result can be extended to any X, assuming the existence of a countable $\succeq -$dense subset of X - this is the Richter-Peleg utility theorem)

2. Clearly this representation is 'worse' than the standard one, in the sense that we cannot recover the preference relation from the utility function. To get round this problem, we can use a **multi-utility representation**. A multi-utility representation of a preference relation $\succeq$ on X is is a set of functions $\mathcal{U}$, where each $u \in \mathcal{U}$ is a function $u : X \to \mathbb{R}$, and these functions represent $\succeq$ in the sense that

$$x \succeq y \text{ if and only if } u(x) \geq u(y) \ \forall \ u \in \mathcal{U}$$

   Show that a multi utility representation has the same information as the original preference relation -i.e. there is a unique preference relation that is consistent with any multi-utility representation

3. One interpretation of the multi utility representation is that each object can be ranked on a number of dimensions, and you are only prepared to say that x is better than y if it is at least as good along all dimensions, and better on one. With that in mind, show how you can construct a multi-utility representation for the partial order $\geq$ on $\mathbb{R}^n$. (i.e. $x \geq y$ if and only if $x_i \geq y_i \ \forall \ i \in \mathbb{N}$)

4. Show that any preference relation on any set X admits a multi utility representation if there is a countable $\succeq -$dense subset of X (hint - you can assume the Richter Peleg Utility theorem. Let $\mathcal{U}$ be the set of all Richter Peleg utility representations. Show that, for any x,y such that neither $x \succeq y$ nor $y \succeq x$ there must be some $u, v \in \mathcal{U}$ such that $u(x) > u(y)$ and $v(y) > v(x)$. Show that this is necessary and sufficient to complete the claim.)

Question 3 Let $\mathcal{P}([0,1])$ be the set of all partitions on $[0,1]$. Show that, if $\succeq$ is a complete preference relation on $\mathcal{P}([0,1])$ such that, if A is a finer partition that B, the $A \succ B$, then $\succeq$ does not have a utility representation.

Question 4 Give an example of a upper semi continuous preference relation that cannot be represented by a continuous utility function