Dynamic Optimization

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Up to this point, we have only considered constrained optimization problems at a single point in time. However, many constrained optimization problems in economics deal not only with the present, but with future time periods as well. We may wish to solve the optimization problem not only today, but for all future periods as well.

The following definitions will be useful: A **control variable** is a variable you can control; for example, you may not be able to control how much capital is in the economy initially, but you can control how much you consume. Things we cannot control completely, but that are nevertheless affected by what we choose as our control are called **state variables**. For example, the amount of capital you have tomorrow depends on the amount you consume today. In optimization problems over time, we want to solve for the control variables at every point of time. The state variables can show up in the objective function or in the constraints, but will be determined by the path of the control variables.

We will look at optimization problems in discrete time and in continuous time, both of which are frequently used in economics. In discrete time problems, we think of time passing in given periods, e.g. a year. A solution will give us a value x_t for the control variable in every time period t . In continuous time problems, we think of time passing continuously. A solution will give us a function (or flow, or stream) $x(t)$ of the control variable over time.

1 Optimization in Discrete Time

You will have to use optimization in discrete time mainly when you are solving life-time consumption problems in Macro. We will therefore look at the standard problem in some detail and use it to outline the general method for solving optimization problems over discrete time.

1.1 Life Time Consumption Problem with Fixed Assets in Discrete Time

Assume an agent has a utility function $u(c_t)$, where c_t is consumption in period t and $u(c_t)$ is a concave function. The agent lives from period 0 until forever and discounts the future at rate $\beta \in (0,1)$. Her life-time utility function is therefore

$$U\left(\{c_t\}_0^\infty\right) = \sum_{t=0}^{\infty} \beta^t u\left(c_t\right).$$

Now suppose she is endowed with initial assets A_0 and does not earn any income during her life. (This is a similar problem to what a retired person would face if she had no income.) The interest rate in the economy is r. Then her budget constraint is

$$A_0 \geq \sum_{t=0}^{\infty} \frac{c_t}{(1+r)^t}.$$

How should the agent choose c_t for each period such that she maximizes her total life time utility and satisfies the budget constraint? In other words, we want to find the solution to

$$\max_{\{c_t\}_0^\infty} U\left(\{c_t\}_0^\infty\right) \text{ subject to } A_0 \geq \sum_{t=0}^{\infty} \frac{c_t}{(1+r)^t}$$

The Lagrangian for this problem can be written as

$$L = \sum_{t=0}^{\infty} \beta^t u(c_t) + \lambda \left(A_0 - \sum_{t=0}^{\infty} \frac{c_t}{(1+r)^t} \right)$$

We know that the constraint will always be binding so long as our utility function exhibits nice properties such as local non-satiation. Therefore we can solve the Lagrangian as if the inequality constraint was an equality constraint.

Unfortunately, this Lagrangian will have an infinite number of first order conditions since t goes to infinity (i.e. there are an infinite number of c_t inputs to the Lagrangian function). This is where a difference equation comes in handy. If we can come up with some sort of condition that must hold between consumption in any two periods, the we can write a difference equation, iterate it, and solve it for all t using an initial condition.

Find the first order conditions of the Lagrangian with respect to an arbitrary c_t and c_{t+1}:

$$\frac{\partial L}{\partial c_t} = \beta^t u'(c_t) - \lambda \frac{1}{(1+r^t)} = 0$$

$$\frac{\partial L}{\partial c_{t+1}} = \beta^{t+1} u'(c_{t+1}) - \lambda \frac{1}{(1+r^{t+1})} = 0$$

Dividing the top equation by the bottom equation we have

$$\frac{u'(c_t)}{u'(c_{t+1})} = (1+r)\beta$$

Say our within period utility function is

$$u(c_t) = \ln(c_t),$$

where sigma is a constant greater than or equal to 0. Our first order condition becomes

$$\frac{c_{t+1}}{c_t} = (1+r)\beta$$

$$c_{t+1} = [(1+r)\beta]\, c_t$$

The solution to this linear difference equation is

$$c_t = c_0 \left[(1+r)\beta\right]^t.$$

Now we have solved our maximization problem for all time periods.

Well, not quite. We don't know what c_0 is. In order to find it, we need to plug this condition into our budget constraint.

$$A_0 = \sum_{t=0}^{\infty} \frac{c_t}{(1+r)^t} = \sum_{t=0}^{\infty} \frac{c_0 \left[(1+r)\beta\right]^t}{(1+r)^t} = \sum_{t=0}^{\infty} c_0 \beta^t = \frac{c_0}{1-\beta}$$

$$c_0 = (1-\beta)A_0$$

Therefore, the solution to the agent's optimization problem is

$$c_t = (1-\beta)\left[(1+r)\beta\right]^t A_0$$

1.2 General Problems

The strategy for solving a general discrete time optimization problem is as follows:

1. Write the proper Lagrangian function.
2. Find the 1st order conditions
3. Solve the resulting difference equations of the control variables
4. Use the constraints to find the initial conditions of the control variables
5. Plug the constraints into the difference equations to solve for the path of the control variable over time

2 Optimization in Continuous Time

Suppose we have a value function $f(x, y)$, where x is a control variable and y is a state variable. We want to control the flow of the value of this function over time so that the lifetime value of the function will be maximized. In other words, we want to find x at every moment t such that

$$\int_{t=0}^{\infty} f(x(t), y(t))dt$$

is maximized subject to constraints. Notice that the maximizer we are looking for is a function itself, $x(t)$. It gives us the time path of the control variable x, not just a particular level of x.

Since time is continuous, the constraint to this problem cannot be a static function. It must tell me the change in my state variable at each point in time, and therefore it must be a differential equation $\dot{y}(t) = F(x(t), y(t))$.

To make things more concrete, we will again look at the basic life time consumption problem that you will encounter in Macro. This is basically the same problem as discussed above, but instead of assuming there are discrete time periods, we now assume that time is continuous. That is, instead of finding consumption c_t for every time period t, we are now looking for a function $c(t)$ that will tell us the level of consumption at every moment of time t.

2.1 Life Time Consumption Problem with Fixed Assets in Continuous Time

Assume an agent has a utility function $u(c(t))$, where $c(t)$ is consumption at time t and $u(c_t)$ is a concave function. The agent lives from period 0 until forever and discounts the future at rate $\beta \in (0, 1)$. Her life-time utility function is therefore

$$U\left(\{c(t)\}_0^\infty\right) = \int_0^\infty e^{-\beta t} u\left(c(t)\right).$$

(Notice how the formula differs for the one in discrete time.) Now suppose the agent is endowed with initial assets A_0 and does not earn any income during her life. The interest rate in the economy is r. Her budget constraint in continuous time can be written as a constraint on the change of her assets and the initial condition:

$$\dot{A}(t) = rA(t) - c(t) \text{ and } A(0) = A_0$$

The question is how the agent should choose the function $c(t)$ to maximize her total life time utility and satisfy the budget constraint. In other words, we want to find the solution to

$$\max_{\{c(t)\}_0^\infty} U\left(\{c(t)\}_0^\infty\right) \text{ subject to } \dot{A}(t) = rA(t) - c(t) \text{ and } A(0) = A_0.$$

To solve optimization problems in continuous time, we abstract from the Lagrangian and use a Hamiltonian. The proof behind why the Hamiltonian works will not be presented in this class, but will be presented in your first semester math class instead.

There are two equivalent formulations of the Hamiltonian; the current value Hamiltonian and the present value Hamiltonian. The current value Hamiltonian for this problem would be expressed as

$$H^c = u(c) + \lambda \dot{A}.$$

Notice this is almost exactly like the Lagrangian function. However, the first order conditions are slightly different:

$$\frac{\partial H}{\partial c} = 0$$

$$\frac{\partial H^c}{\partial \lambda} = \dot{A}$$

$$\frac{\partial H^c}{\partial A} = \beta\lambda - \dot{\lambda}$$

We would need to solve this system using our analysis from differential equations.

The present value Hamiltonian would be formulated this way:

$$H^p = e^{-\beta t} H^c = e^{-\beta t} u(c) + \mu A.$$

Notice the objective function is now discounted in the Hamiltonian, whereas before it was not. The first order conditions are

$$\frac{\partial H^p}{\partial c} = 0$$

$$\frac{\partial H^p}{\partial \mu} = \dot{A}$$

$$\frac{\partial H^p}{\partial A} = -\dot{\mu}$$

Let's look at an example where we have a utility function given.

Example: $\max \int_0^\infty e^{-\beta t} \ln\left[c_t\right] dt$ subject to $\dot{A} = rA - c$. Assume that A_0 is known.

We form the current value Hamiltonian

$$H = \ln(c_t) + \lambda\left(rA - c\right)$$

The first order conditions are

$$\frac{\partial H}{\partial c} = \frac{1}{c} - \lambda = 0$$

$$\frac{\partial H}{\partial \lambda} = ra - c = \dot{A}$$

$$\frac{\partial H}{\partial A} = \lambda r = \beta\lambda - \dot{\lambda}$$

From the first condition, we get $\ln(c_t) = -\ln(\lambda_t)$. Taking the derivative of both sides of this function, we get

$$\frac{\dot{c}}{c} = -\frac{\dot{\lambda}}{\lambda}.$$

From the third condition we get

$$\beta - r = \frac{\dot{\lambda}}{\lambda}$$

Setting these two conditions equal to each other we get

$$\frac{\dot{c}}{c} = r - \beta$$

The differential equation which solves this is $c_t = c_0 e^{(r-\beta)t}$. Now it remains to find the initial condition for c_0, which we can find using the budget constraint. We know that the present discounted value of our consumption must equal our initial assets, so

$$A_0 = \int_0^\infty e^{-rt} c(t) dt = \int_0^\infty e^{-rt} c_0 e^{(r-\beta)t} dt = c_0 \int_0^\infty e^{-\beta t} = c_0 \frac{1}{\beta}$$

$$c_0 = \beta A_0$$

Therefore, the solution is

$$c_t = \beta A_0 e^{(r-\beta)t}$$

2.2 General Problems

The strategy for solving a general continuous time optimization problem is as follows:

1. Write the Hamiltonian.
2. Find the first order conditions.
3. Obtain differential equations in control and state variables.
4. Solve one of them.
5. Use the budget constraint to find the initial conditions.

3 Homework

Go over the examples presented in these notes and make sure you understand every step.