Dynamical Systems Solutions[1]

Math Camp 2012

1 Single Differential Equations

1. Find the general solutions to the following equations.

   (a) $\dot{y} - 2y = 1$

   We have that $p(t) = -2$, $q(t) = 1$ therefore $H(t) = -2t + C$, then

   $$y(t) = -\frac{1}{2} + \alpha e^{2t}$$

   (b) $2\dot{y} + 5y = 2$

   Then

   $$\dot{y} = -\frac{5}{2}y + 1 \Leftrightarrow y(t) = \frac{2}{5} + \alpha e^{-\frac{5}{2}t}$$

   (c) $\dot{y} - 2y = 1 - 2x$ (here x is used instead of t)

   We have that $p(t) = -2$, $q(t) = 1 - 2t$ therefore $H(t) = -2t + C$, then

   $$y(t) = e^{2t-C}\left(\int (1-2t)e^{-2t+C}dt + D\right) = t + \gamma e^{2t}$$

   (d) $x\dot{y} - 4y = -2nx$ (here x is used instead of t)

   First of all notice that at $t = 0$, $y(0) = 0$. When $t \neq 0$ we have that $\dot{y} = \frac{4}{t}y - 2n$ $p(t) = -\frac{4}{t}$, $q(t) = -2n$ therefore $H(t) = -4\ln(|t|) + C$, then

   $$y(t) = e^{4\ln(|t|)-C}\left(\int -2ne^{-4\ln(|t|)+C}dt + D\right) = t^4e^C\left(\int -2ne^Ct^{-4}dt + D\right) = \frac{2}{3}nt + \gamma t^4$$

   (e) $\dot{y} = e^x y$

   We have that $p(t) = -e^t$, $q(t) = 0$ therefore $H(t) = -e^t + C$, then

   $$y(t) = e^{e^t - C}\left(\int 0dt + D\right) = \gamma e^{e^t}$$

2. Find the general solutions to the previous questions given that $y_0 = 1$.

   (a) $y(0) = -\frac{1}{2} + \alpha = 1$ therefore $\alpha = \frac{3}{2}$

   (b) $y(0) = \frac{2}{5} + \alpha = 1$ therefore $\alpha = \frac{3}{5}$

   (c) $y(0) = \gamma = 1$

   (d) $y(0) = 0$ cannot be 1

   (e) $y(0) = e\gamma = 1$ therefore $\gamma = e^{-1}$

[1] If you find any typo please email me: Maria_Jose_Boccardi@Brown.edu

2 Difference equation

Solve the difference equation $y_{t+1} = by_t^2$. Find the steady states. Graph the function with respect to time. Draw the phase diagrams. Determine stability of the steady states. (Hint: there will be several cases.) What about $by_{t+1} = ay_t^2 + c$? (This is complicated, so think about it graphically for starters. Don't worry if you can't finish the whole problem with all the cases.)

So we have the difference equation $y_{t+1} = by_t^2$, and a given y_0. Therefore we have that:

$y_1 = by_0^2$, then

$y_2 = by_1^2 = b(by_0^2)^2 = b^3y_0^4$, then

$y_3 = by_2^2 = b(b^3y_0^4)^2 = b^7y_0^8$, then

$y_4 = by_3^2 = b(b^7y_0^8)^2 = b^{15}y_0^{16}$, therefore $y_t = b^{t^2-1}y_0^{t^2}$

The steady state of this difference equation is y such that $y_{t+1} = y_t$, or equivalently $\Delta y_t = 0$, therefore we should have that $y_t = by_t^2$, that is (assuming $b \neq 0$)

$$y_{t+1} = y_t \Leftrightarrow y_t = by_t^2 \Leftrightarrow y_t - by_t^2 = 0 \Leftrightarrow y(t)\,(1 - by(t)) = 0$$

, that is we have two steady states $y^{(1)} = 0$ and $y^{(2)} = \frac{1}{b}$. If $b = 0$ then after y_0, $y_t = 0$ for all $t > 0$.

The stability of the steady state would be given by the value of the derivative of y_{t+1} wrt y_t valuated at the steady state. That is the condition for stability of $y^{(1)}$ is that

$$\left|\frac{\partial y_{t+1}(y^*)}{\partial y_t}\right| < 1 \Leftrightarrow |2by^*| < 1 \Leftrightarrow |0| < 1$$

which is satisfied. On the other hand, the condition for the stability of the $y^{(2)}$ is given by

$$\left|\frac{\partial y_{t+1}(y^*)}{\partial y_t}\right| < 1 \Leftrightarrow \left|2b\frac{1}{b}\right| < 1 \Leftrightarrow |2| < 1$$

, which is not satisfied, therefore this steady state is not stable. In the case where $b = 0$ we have that $\frac{\partial y_{t+1}(y^*)}{\partial y_t} = 0$ for all y therefore it is stable.

In the case where $by_{t+1} = ay_t^2 + c$ we have that if $b \neq 0$ then $y_{t+1} = \frac{a}{b}y_t^2 + \frac{c}{b}$ and then given y_0, when $c \neq 0$ there is not a nice solution to the iteration. Anyway, we can find the steady state and analyze stability in a pretty simple way. We know that the steady state should be such that $y_t = by_t^2$, that is (assuming $b \neq 0$)

$$by_t = ay_t^2 + c \Leftrightarrow ay_t^2 - by(t) + c = 0 \Leftrightarrow y(t) = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}$$

In order to analyze stability we should check the first derivative, where the condition is given by

$$\left|\frac{\partial y_{t+1}(y^*)}{\partial y_t}\right| < 1 \Leftrightarrow \left|2\frac{a}{b}y^*\right| < 1 \Leftrightarrow \left|2\frac{a}{b}\frac{b \pm \sqrt{b^2 - 4ac}}{2a}\right| < 1 \Leftrightarrow \left|1 \pm \frac{\sqrt{b^2 - 4ac}}{b}\right| < 1$$

therefore we have that if $b > 0$ then $y(t) = \frac{b+\sqrt{b^2-4ac}}{2a}$ is not going to be stable; whereas $y(t) = \frac{b-\sqrt{b^2-4ac}}{2a}$ would depend on whether $\frac{\sqrt{b^2-4ac}}{b} < 2$ that is

$$\sqrt{b^2 - 4ac} < 2b \Leftrightarrow b^2 - 4ac < 4b^2 \Leftrightarrow -4ac < 3b^2 \Leftrightarrow -\frac{4ac}{3} < b^2$$