FIXED POINT THEOREMS
Econ 2010 - Fall 2013

Fixed point theory serves as an essential tool for various branches of mathematical analysis and its applications. Loosely speaking, there are three main approaches in this theory: the metric, the topological and the order-theoretic approach, where representative examples of these are: Banach's, Brouwer's and Tarski's theorems respectively.

1 Metric Approach

Definition 1 (Fixed point property) *A metric space X is said to have the fixed point property if every continuous self-map on X has a fixed point.*

1.1 Banach Fixed Point Theorem

As it was stated and proved in the class notes, we have that in a complete metric space a contraction must map a point to itself, that is it must have a fixed point, and even more, it is unique.

Theorem 2 (Banach's Fixed Point Theorem) *Let X be a complete metric space, and f be a contraction on X. Then there exists a unique x^* such that $f(x^*) = x^*$.*

Definition 3 (Contraction) *Let X be a metric space, and $f : X \to X$. We will say that f is a contraction if there exists some $0 < k < 1$ such that $d(f(x), f(x^*)) \leqslant kd(x, y)$ for all $x, y \in X$. The inf of such k's is called the contraction coefficient.*

The Banach Fixed Point theorem is also called the contraction mapping theorem, and it is in general use to prove that an unique solution to a given equation exists. There are several examples of where Banach Fixed Point theorem can be used in Economics *for more detail you can check Ok's book, Chapter C, part 7* For concreteness purposes let focus in one of the most known applications for Banach's theorem for economists, *Bellman's functional equations.*

1.1.1 Application

I.- Dynamic Optimization

We have the following problem

$$\max_{(x^m)} \varphi(x^0, x^1) + \sum_{i=1}^{\infty} \delta^i \varphi(x^i, x^{i+1})$$

such that

$$x^1 \in \Gamma(x^0) \quad \text{and} \quad x^{m+1} \in \Gamma(x^m) \text{ for } m = 1, 2, ..$$

Where if we were to assume that φ is continuous and bounded and Γ is compact valued and continuous, we can prove (*we are not going to do it now*) that if X is a nonempty convex subset of Euclidean space then this problem has an unique solution for each initial value.

II. Differential equations

One of the very important theoretical application of Banach fixed point theorem is the proof of existence and uniqueness of solutions of differential equations sufficiently regular. In this application, the complete metric space K is a set of functions, and the map F transforms a function into another function (we often say that F is an operator). The trick is to show that a solution of the differential equation, if its exists, is a fixed point of the operator F.

Consider for example the case of

$$y' = e^{-x^2}$$

the solution is given by

$$y = \int e^{-x^2} dx$$

1.2 Caristi Fixed Point Theorem

Theorem 4 *Let Φ be a self-map on a complete metric space X. If $d(x, \Phi(x)) \leqslant \varphi(x) - \varphi(\Phi(x))$ for all $x \in X$, for some lower semicontinuous $\varphi \in \mathbb{R}^X$ that is bounded from below, then Φ has a fixed point in X.*

This theorem is a generalization of the Banach fixed point theorem, in particular if $\Phi \in X^X$ is a contraction with the contraction coefficient $K \in (0,1)$ then the hypothesis of Caristi's theorem is satisfied for $\varphi \in \mathbb{R}_+^X$ defined for $\varphi(x) \doteqdot \frac{1}{1-K} d(x, \Phi(x))$, where φ is continuous and $\varphi(x) - \varphi(\Phi(x)) > d(x, \Phi(x))$. This is due to the fact that:

$$\begin{aligned} \varphi(x) - \varphi(\Phi(x)) &= \frac{1}{1-K}\left[d(x, \Phi(x)) - d(\Phi(x), \Phi^2(x))\right] \\ &> \frac{1}{1-K}\left[d(x, \Phi(x)) - K d(x, \Phi(x))\right] \\ &= d(x, \Phi(x)) \end{aligned}$$

But the generalization is substantial when we observe that Caristi doesn't require the involved self-map to have the contraction property and not even that the self-map being continuous.

2 TOPOLOGICAL APPROACH

2.1 Brouwer's Fixed Point Theorem

Theorem 5 (Brouwer's Fixed Point Theorem for the Unit Ball) *B^n has the fixed point property, n=1,2,...*

A good number of fixed point theorems that are invoked in certain parts of economic theory can be derived by using Brouwer's fixed point theorem for the Unit Ball.

Theorem 6 (Brouwer's Fixed point Theorem) *For any given $n \in \mathbb{N}$, let S be nonempty, closed, bounded and convex subset of $\mathbb{R}^n$. If Φ is a continuous self-map on S, then there exists an $x \in S$ such that $\Phi(x) = x$.*

2.1.1 Applications

From Brouwer's theorem we can extend to new Fixed Point theorems in the following way

Proposition 1 *For any $n \in \mathbb{N}$, any continuous $\Phi : B^n \to \mathbb{R}^n$ with $\Phi(S^{n-1}) \subseteq B^n$, has a fixed point.*

In particular if we take any continuous $\Phi : B^n \to \mathbb{R}^n$ we can define

$$r(x) \equiv \begin{matrix} x & if & x \in B^n \\ \frac{x}{d_2(x,0)} & & otherwise \end{matrix}$$

, where $ro\Phi$ is a continuous self-map on B^n, and then Brouwer's apply and we have that $r(\Phi(x)) = x$ for some $x \in B^n$.

This application in particular is use to demonstrate the existence of a Walrasian equilibrium, in particular Kakutani's extension that is presented in the following section.

Other application of the theorem is to prove that every strictly positive nxn matrix has a positive eigenvalue and a positive eigenvector.

2.2 Kakutani's Fixed Point Theorem

Kakutani's theorem is a famous generalization of Brouwer theorem.

Theorem 7 *For any given $n \in \mathbb{N}$, let X be a nonempty, closed, bounded and convex subset of $\mathbb{R}^n$. If Γ is a convex-value self-correspondence on X that has a closed graph, then Γ has a fixed point, that is, there exists an $x \in X$ with $x \in \Gamma(x)$.*

Where the requirement of a closed graph can be replaced with upper hemicontinuity when Γ is closed-valued.

2.2.1 Applications

Nash Equilibrium

Definition 8 *Let $\mathcal{G} \doteqdot \left\{(X_i, \pi_i)_{i=1,\dots,m}\right\}$ be a strategic game. We say that $x^* \in X$ is a Nash equilibrium if $x^* \in \arg\max\left\{\pi_i(x_i, x^*_{-i}) : x_i \in X_i\right\}$ for all i=1,...,m. A Nash equilibrium x^* is said to be symmetric if $x^*_1 = ... = x^*_m$. We denote the set of all Nash and symmetric Nash equilibria of a game $\mathcal{G}$ by $NE(\mathcal{G})$ and $NE_{sym}(\mathcal{G})$ respectively*

Definition 9 *If each X_i is a nonempty compact subset of a Euclidean space, then we say that the strategic game $\mathcal{G} \doteqdot \left\{(X_i, \pi_i)_{i=1,\dots,m}\right\}$ is a* ***compact Euclidean game.*** *If in addition $\pi_i \in C(X)$ for each i=1,..m we say that $\mathcal{G}$ is a* ***continuous and compact Euclidean game.*** *If instead, each X_i is convex and compact, and each $\pi_i(x_{-i})$ is quasiconcave for any given $x_{-i} \in X_{-i}$ then $\mathcal{G}$ is called a* ***convex and compact Euclidean game.*** *Finally, a compact Euclidean game which is both convex and continuous is called a* ***regular Euclidean game.***

Theorem 10 (Nash's Existence Theorem) *If* $\mathcal{G} \doteqdot \left\{(X_i, \pi_i)_{i=1,\dots,m}\right\}$ *is a regular Euclidean game, then* $NE(\mathcal{G}) \neq \varnothing$

Where the result follows from Kakutani's theorem defining the **best response correspondence** as $b_i(x_{-i}) \doteqdot \arg\max\{\pi_i(x_i, x_{-i}) : x_i \in X_i\}$ and $b(x) \doteqdot \prod_{i=1}^{m} b_i(x_{-i})$. By Weierstrass b is well defined. Note that if $x \in b(x)$, then $x_i \in b(x_{-i})$ for all i then $x \in NE(\mathcal{G})$, so we can show existence by Kakutani's.

Existence of Walrasian Equilibrium

Proposition 2 (17.C.1: (MWG)) *Suppose that* $z(p)$ *is a function defined for all strictly positive price vectors* $p \in \mathbb{R}^L_{++}$*, and satisfying: {(i)* $z()$ *is continuous, (ii)* $z()$ *is homogenous of degree zero, (iii)* $pz(p) = 0$ *for all* p*; (iv)* $\exists s > 0$ *s.t.* $z_l(p) > -s$ *for all* l, p*; and (v) if* $p^n \to p$*, where p*$\neq 0$ *and* $p_l = 0$ *for some* l*, then* $\max\{z_1(p^n), \dots, z_L(p^n)\} \to \infty$*} then the system of equations* $z(p) = 0$ *has a solution. Hence* ***a Walrasian equilibrium exists in any pure exchange economy in which*** $\sum \omega_i >> 0$ ***and very consumer has continuous, strictly convex and strongly monotone preferences.***

I am not going to go into the details of the demonstration since you are going to do it in Microeconomics class. Just note that given homogeneity of degree zero (condition ii) of z, we can normalize prices and restrict the problem to the following simplex $\Delta = \left\{p \in \mathbb{R}^L_+ : \sum_l p_l = 1\right\}$, but the function is well defined only for prices in the following set $Interior\Delta = \{p \in \Delta : p_l > 0 \text{ for all } l\}$. The idea of the proof is to construct a correspondence f from Δ to Δ, argue that the fixed point of this correspondence is when $p^* \in f(p^*)$ which in turn implies that $z(p^*)$. In this proof

$$
\begin{aligned}
f(p) &\doteqdot \frac{p + z^+(p)}{\alpha(p)} \\
&\quad \text{where} \\
z_l^+(p) &\doteqdot \max\{0, z_l(p)\} \\
\alpha(p) &\doteqdot \sum_l \left[p_l + z_l^+(p)\right]
\end{aligned}
$$

Therefore if we prove that f is convex valued and upper hemicontinuous (or closed graph) we can apply Kakutani's to show that p^* exists.

2.3 ORDER THEORETICAL APPROACH

2.4 Tarsi's Fixed Point Theorem

First, some introductory definitions regarding Order theory.

Definition 11 (Classes of preference relations) *(I) If* R *is reflexive, symmetric and transitive it is an* ***equivalence relation***

(II) If R *is reflexive and transitive then R is a* ***preorder*** *[preference relation]*

(III) If R is reflexive, transitive and antisymmetric then R is a ***partial order***

(IV) A complete partial order is a ***linear order***

Definition 12 (Poset/Loset) *We call (X, R) a poset if R is a partial order, and a loset if R is a linear order.*

Definition 13 (Maximal Element) *Let $\succcurlyeq$ be a binary relation on X, the maximal elements of X according to $\succcurlyeq$ are defined as $Max(X, \succcurlyeq) = \{x \in X/y \succ x$ for no $y \in X\}$*

Definition 14 (Conditional complete poset) *Let $(X, \succcurlyeq)$ be a poset. We say that (X,$\succcurlyeq$) is a conditionally complete if $\succcurlyeq - \sup$ exists for every nonempty subset S of X such that $x \succcurlyeq S$ holds for at least one x in X. That is a poset (X,$\succcurlyeq$) is conditionally complete iff any nonempty subset S of X with an $\succcurlyeq -$upper bound in X has a $\succcurlyeq$-supremum in X.*

Definition 15 (Lattice) *Let $(X, \succcurlyeq)$ be a poset. If $x \bigvee y$ and $x \bigwedge y$ exists for every $x, y \in X$, we say that (X,$\succcurlyeq$) is a lattice. If $\bigvee S$ and $\bigwedge S$ exist for every subset S of X, we then say that (X,$\succcurlyeq$) is a* ***complete lattice.***

Theorem 16 (Tarski's FPT) *Let $(X, \succcurlyeq)$ be a conditionally complete poset with a $\succcurlyeq -$ minimum and $\succcurlyeq -$maximum. Then for every $\succcurlyeq -$preserving self-map f on X, $Fix(f)$ contains a $\succcurlyeq$-minimum and $\succcurlyeq$-maximum element.*

The following is a consequence of Tarski's fixed point theorem, which says that the fixed point set of an order preserving self-map on a complete lattice, inherits the lattice structure of that poset, that is, it is a complete lattice itself under the original partial order.

Theorem 17 (Knaster-Tarski's Theorem) *Let $(X, \succcurlyeq)$ be a complete lattice and f a Let $\succcurlyeq -$ preserving self-map on X. Then $(Fix(f), \succcurlyeq)$ is a complete lattice.*

2.4.1 Applications

Example 18 *For any positive integer n, let a and b be two n-vector with $a \leqslant b$ and define $I \doteqdot [a_1, b_1] \times ... \times [a_n, b_n]$. Let $f_i : I \to \mathbb{R}$ be an increasing function and the set $\mathbf{f} : \doteqdot (f_1, ..., f_n)$. As $(I, \geqslant)$ is a complete lattice and f is obviously $\geqslant -$preserving the Knaster-Tarski theorem says that f is sure to have a fixed point, provided that $a \leqslant f(a) \leqslant f(b) \leqslant b$*

Corollary 19 *A corollary of the theorem is that every increasing self-map on $[0, 1]$ has a fixed point.*