# G5212: Game Theory

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# Static Bayesian Games - Applications

- Now we have a sense of how to use Bayesian Games to model strategic situations with imperfect information
- We will now look at two classic applications
    1. Auctions
    2. Global Games

# Auctions - Introduction

- The design of auctions is a huge area in the applied and theoretical micro literature
- Basic idea
  - One entity has some objects to sell
  - A set (usually more than 1) of potential buyers
  - Buyers submit bids
  - Based on these bids the auction design determines
    1. Who gets the object
    2. Who pays what

# Auctions - Introduction

- Many variants in the set up
  - Who knows what?
  - Private value vs common value?
  - Single good or bundles?
- Huge number of choices when it comes to auction design
  - First price vs second price
  - Dynamic vs Static
  - All pay
  - Reserve prices

# Auctions - Introduction

- Lots of questions: How do different mechanisms affect
  - Profit
  - Welfare/Efficiency
  - Ability to collude
- Both in theory and in practice
- Lots of applications (worth billions of dollars)
  - Ebay
  - Wireless Spectrum Auction
  - Treasury bills
  - Art market

# Auctions - Introduction

- We will focus on two of the simplest cases
- Independent private value
  - Sealed Bid Second Price Auction
  - Sealed Bid First Price Auction
- Revenue Equivalence
- If you are interested take Advanced Micro in the summer!

# Independent Private Value Set Up

- This Independent Private Value set up is described as follows
  - A seller has one indivisible object.
  - There are $n$ bidders whose valuations for the object are drawn independently from a continuous distribution $F$ on $[v_*, v^*]$
  - Valuations private information

# Second Price Sealed Bid Auction

- Second Price Sealed Bid Auction is a mechanism for determining who gets what
  - The bidders simultaneously submit bids $s_i \in [0, \infty)$.
  - The highest bidder wins the object and pays the *second highest* bid.
  - Given a profile of bids, $s$, let $W(s) \equiv \{k : \forall j, s_k \geq s_j\}$ be the set of highest bidders.
  - Then the game is simply the following:

$$u_i(s_i, s_{-i}) = \begin{cases} v_i - \max_{j \neq i} s_j & \text{if } s_i > \max_{j \neq i} s_j \\ \frac{1}{|W(s)|}(v_i - s_i) & \text{if } s_i = \max_{j \neq i} s_j \\ 0 & \text{if } s_i < \max_{j \neq i} s_j. \end{cases}$$

# Second Price Sealed Bid Auction

- What strategies form an equilibrium in this game
    - Reminder: Strategy is a mapping from types to bids
- Claim: It is weakly dominant to bid one's own valuation
- Why?

# Second Price Sealed Bid Auction

- Consider the player with value $v$ who bids $s > v$
- Compare this to the strategy of bidding $v$
- For any strategy $s_j : [v_*, v^*] \to \mathbb{R}_+$ of the other players, and for any set of values $v_j$ for the other players the only important metric is

$$\bar{s} = \max_{j \neq i} s_j(v_j)$$

# Second Price Sealed Bid Auction

- If $\bar{s} \leq v$ then bidding $s$ and $v$ give the same payoff
  - win the item and pay $\bar{s}$
- If $\bar{s} > s$ then bidding $s$ and $v$ give the same payoff
  - Do not win the item and get 0
- Only difference is if $\bar{s} \in (v, \bar{s}]$
  - In this case bidding $v$ means a payoff of 0
  - Bidding $s$ means winning the object but paying $\bar{s} > v$
  - Negative payoff
- Similar logic shows that bidding $v$ weakly dominates bidding $s < v$

# Second Price Sealed Bid Auction

- In a Second Price Sealed Bid Auction it is a symmetric BNE for every player to have a bidding strategy

$$s_i(v_i) = v_i$$

- Easy to show - as this strategy weakly dominates all others for each player and each type, it is a best response any strategy of any other player
- Is this the only BNE?

# Second Price Sealed Bid Auction

- Consider an auction with two players
- Player 1 always bids $v_*$ regardless of type
- Player 2 always bids $v^*$ regardless of type
- Payoff or player 1 is 0
- Payoff of player 2 is $v - v_*$
- Can either player do better by deviating?

# Second Price Sealed Bid Auction

- No!
- In order to win the auction, player 1 would have to bid $v > v^*$, guaranteeing negative payoff
- If player two deviated to a lower bid, would still always win the auction and pay $v_*$

# First Price Sealed Bid Auction

- First Price Sealed Bid Auction is a different mechanism
  - The bidders simultaneously submit bids $s_i \in [0, \infty)$.
  - The highest bidder wins the object and pays their bid.
  - Given a profile of bids, $s$, let $W(s) \equiv \{k : \forall j, s_k \geq s_j\}$ be the set of highest bidders.
  - Then the game is simply the following:

$$u_i(s_i, s_{-i}) = \begin{cases} v_i - s_i & \text{if } s_i > \max_{j \neq i} s_j \\ \frac{1}{|W(s)|}(v_i - s_i) & \text{if } s_i = \max_{j \neq i} s_j \\ 0 & \text{if } s_i < \max_{j \neq i} s_j . \end{cases}$$

# First Price Sealed Bid Auction

- Is it optimal to bid your value in a first price auction?
- No!
- This strategy guarantees a payoff of 0
- Whereas (for example) bidding half your value will provide a positive payoff on average
- Want to 'shade down' your bid
- But by how much?
    - Benefit: get more when you win the auction
    - Cost: lower chance of winning

# First Price Sealed Bid Auction

- Optimization problem for person with value $v$

$$\max_{s}(v - s)P(\bar{s} < s)$$

- Lets make things a little simpler
- Assume that values are uniformly distributed on $[0, 1]$
- Guess and verify that everyone uses a linear bidding strategy $s = \theta v$

# First Price Sealed Bid Auction

- Optimization problem becomes

$$\begin{aligned}
\max_s (v - s^*) P(s_1 &\leq s^*, s_2 \leq s^*, ...) \\
&= \max_s (v - s^*) P(\theta v_1 \leq s^*, \theta v_2 \leq s^*, ...) \\
&= \max_s (v - s^*) P(v_1 \leq \frac{s^*}{\theta}, v_2 \leq \frac{s^*}{\theta}, ...) \\
&= \max_s (v - s^*) \left(\frac{s^*}{\theta}\right)^{n-1}
\end{aligned}$$

# First Price Sealed Bid Auction

- Gives FOC

$$
\begin{aligned}
\frac{(n-1)(v-s^*)}{\theta}\left(\frac{s^*}{\theta}\right)^{n-2}-\left(\frac{s^*}{\theta}\right)^{n-1} &= 0 \\
\frac{(n-1)(v-s^*)}{\theta} &= \frac{s^*}{\theta} \\
s^* &= \frac{n-1}{n}v
\end{aligned}
$$

- Verifies linear bid strategy
- Bid 'shaded down' by $\frac{n-1}{n}$
  - when $n = 2$, bid half your value
  - as $n \to \infty$ bid approaches value

# First Price Sealed Bid Auction

- Lets think about a more general case
- Valuations not necessarily distributed uniformly

$$
\begin{aligned}
(v - s^*)P(\bar{s} &< s^*) \\
&= (v - s^*)P(s_1 \leq s^*, s_2 \leq s^*, \ldots) \\
&= (v - s^*)P(s(v_1) \leq s^*, s(v_2) \leq s^*, \ldots) \\
&= (v - s^*)P(v_1 \leq s^{-1}(s^*), v_2 \leq s^{-1}(s^*), \ldots) \\
&= (v - s^*)F\left(s^{-1}(s^*)\right)^{n-1}
\end{aligned}
$$

- Assume that solution is a function is a bid strategy which is strictly increasing and continuous in $v$

# First Price Sealed Bid Auction

- First order conditions

$$(v-s^*)(n-1)f(s^{-1}(s^*))s^{-1\prime}(s^*)F\left(s^{-1}(s^*)\right)^{n-2}-F\left(s^{-1}(s^*)\right)^{n-1}$$

$$= 0$$

$$\Rightarrow F\left(s^{-1}(s^*)\right) = \frac{(v-s^*)(n-1)f(s^{-1}(s^*))}{s'(s^{-1}(s^*))}$$

- For symmetric equilibrium $s^{-1}(s^*) = v$

$$F(v) = \frac{(v-s(v))(n-1)f(v)}{s'(v)}$$

- Where we replace $s^*$ with $s(v)$

# First Price Sealed Bid Auction

- This is a differential equation in $v$
- Can be solved using the initial condition that $s(v_*) = v_*$

$$s(v) = v - \frac{\int_{v_*}^{v} F^{n-1}(x)dx}{F^{n-1}(v)}$$

# Comparing Revenue

- So we now have two different auction mechanisms
- And have figured out the optimal bidding procedure in each case
- If you wanted to maximize revenue, which would you choose?

# Comparing Revenue

- To make things simple, let's concentrate on the uniform distribution case.
- Type $v$ bids

$$\frac{n-1}{n}v$$

- So expected revenue is

$$\begin{aligned} & \frac{n-1}{n}E\left\{\max\left(v_1,...v_n\right)\right\} \\ = & \frac{n-1}{n}\int_0^1 vf_{(1)}(v)dv \end{aligned}$$

- where $f_{(1)}(v)$ is the 1st order statistic of the possible values
  - i.e. the distribution of the maximal value of $n$ draws from $U[0,1]$
  - We can look this up:

$$f_{(1)}(v) = nv^{n-1}$$

# Comparing Revenue

- Plugging in gives

$$\begin{aligned} &= \frac{n-1}{n}\int_0^1 vnv^{n-1}dv \\ &= (n-1)\int_0^1 v^n dv \\ &= \frac{(n-1)}{(n+1)} \end{aligned}$$

# Comparing Revenue

- How about the second price revenue
- Here, everyone bids their value, but we get the second highest bid
- So expected revenue is

$$\int_0^1 v f_{(2)}(v) dv$$

- where $f_{(2)}(v)$ is the 2nd order statistic of the possible values
  - i.e. the distribution of the second highest value of $n$ draws from $U[0,1]$
  - We can look this up:

$$f_{(1)}(v) = n(n-1)(v^{n-2} - v^{n-1})$$

# Comparing Revenue

- Plugging in

$$\begin{aligned} & \int_0^1 vn(n-1)(v^{n-2} - v^{n-1})dv \\ = \quad & n(n-1)\left[\int_0^1 v^{n-1}dv - \int_0^1 v^n dv\right] \\ = \quad & n(n-1)\left[\frac{1}{n} - \frac{1}{n+1}\right] = \frac{n-1}{n+1} \end{aligned}$$

- Magic!

# Revenue Equivalence Theorem

- Is this something special about the particular set up we chose?
- No!

**Theorem (Revenue Equivalence)**

*For an auction amongst $n$ bidders the values of which are independently and identically distributed with some continuous distribution $F$, any mechanism which*

1. *Always awards the object to the bidder with the highest value*
2. *Gives bidder with valuation $v_*$ zero profits*

*Generates the same revenue in expectation*

# Coordination Game

- Consider the following situation:
- Two investors are deciding whether to attack a currency
- If they both attack they will move the currency
- But there is a cost to being the only person to attach

# A Global Game

| | In | Out |
|---|---|---|
| In | $\theta, \theta$ | $\theta - t, 0$ |
| Out | $0, \theta - t$ | $0, 0$ |

- Players choose 'in' or 'out'
  - $\theta$ value of coordinating action (state of nature)
  - $t$ cost of failure to coordinate
- Full information case: if $0 \leq \theta \leq t$ two equilibria
  - {In, In} (Payoff Dominant)
  - {Out, Out} (Risk Dominant)

# A Global Game

| | In | Out |
|---|---|---|
| In | $\theta, \theta$ | $\theta - t, 0$ |
| Out | $0, \theta - t$ | $0, 0$ |

- What if there is uncertainty about $\theta$? [Carlsson and van Damme 1993]
  - Each agent receives an independent signal $z = \theta + \varepsilon$
- Equilibrium a pair of threshold strategies $(\bar{z}, \bar{z})$
  - As variance of $\varepsilon$ goes to zero, $\bar{z} \to \frac{t}{2}$
- For small variance, coordination is 'difficult': for $0 < z < \frac{t}{2}$
  - {In, In} is 'almost certainly' an equilibrium for underlying game
  - But 'out' is unique equilibrium strategy

# A Global Game

| | In | Out |
|---|---|---|
| In | $\theta, \theta$ | $\theta - t, 0$ |
| Out | $0, \theta - t$ | $0, 0$ |

- Intuition: lack of common knowledge leads to unravelling
  - Say threshold is $\bar{z}$
  - If I received signal of exactly $z$, then there is exactly a $\frac{1}{2}$ chance that the other player will choose in
  - Payoff of playing 'in' is

$$E(\theta|\bar{z}) - \frac{t}{2}$$

  - As the variance of $\varepsilon$ goes to zero, this becomes negative for any $\bar{z} < \frac{t}{2}$