G5212: Game Theory

Mark Dean

Spring 2017

Nash Equilibrium

- We now have one way of solving (i.e make predictions) in games
  - We assume common knowledge of rationality
  - This allows us to focus on the set of strategies which are rationalizable/survive IDSDS
- But this may be unsatisfactory for two reasons

1. Predictions may not be very tight
   - Think matching pennies
2. We may find it unrealistic to assume that people have beliefs which are wrong

Nash Equilibrium

- Our next refinement is therefore to put further restrictions on beliefs
- In particular we will demand that the beliefs μ that we introduced last time are consistent with play in the game
  - Allows us to make more precise predictions
  - Insists that beliefs are correct
- We will insist
  1. Players best respond to beliefs
  2. Beliefs are generated by the play of other players - i.e. for mixed strategies σ_j^*

$$\mu_i = \prod_{j \neq i} \sigma_j^*$$

- Note, that this means that we can drop beliefs from the definition

Nash Equilibrium in Pure Strategies

Definition

A Nash equilibrium in pure strategies is a strategy profile $(s_1^*, ..., s_n^*)$ such that, for all i, for all $s_i \in \Delta S_i$, $u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*)$.

- Each player is doing the best thing, given what others are doing
- Note: A Nash equilibrium **always assigns a strategy to each player!**
  - If I ask for a Nash equilibrium, and you do not give me a strategy for each player, then you have done something wrong

Prisoner's Dilemma

Example

Prisoner's Dilemma

		Bob: Confess	Bob: Don't Confess
Anne	Confess	$-6, -6$	$0, -9$
	Don't Confess	$-9, 0$	$-1, -1$

- C, C is unique Nash Equilibrium in Pure Strategies

Bach or Stravinsky

Example

Prisoner's Dilemma

		Bob	
		Bach	Stravinsky
Anne	Bach	$2, 1$	$0, 0$
	Stravinsky	$0, 0$	$1, 2$

- Anne and Bob are picking a concert to go to
- Both prefer to go together than to go alone
- Anne prefers Bach while Bob prefers Stravinsky

- B, B and S, S are the two N.E. in pure strategies

A Game With No Name

Example

A Game with No Name

		Bob			
		X	Y	Z	W
	A	5, 2	3, 6	0, 2	2, 2
Anne	B	7, 6	7, 7	3, 2	2, 0
	C	3, 2	2, 3	3, 1	2, 0
	D	2, 1	9, 2	2, 0	2, 1

- D, Y is the NE in pure strategies

Matching Pennies

Example

Matching Pennies

		Bob H	Bob T
Anne	H	$+1, -1$	$-1, +1$
	T	$-1, +1$	$+1, -1$

- No NE in pure strategies
- Note this shows we can have strategies which are rationalizable, but not part of a NE
- Yikes! Economists like
  - Existence
  - Uniquence
- Maybe NE is not useful if we can't guarantee existence

Nash Equilibrium in Mixed Strategies

- Maybe we can get further if we allow for **mixed** strategies?

Definition

A Nash equilibrium is a strategy profile $(\sigma_1^*, ..., \sigma_n^*)$ such that, for all i, for all $\sigma_i \in \Delta(S_i)$, $u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*)$.

- Again, this is based in the idea that each player is doing the best thing given what everyone else is doing
- Let's make this formal
  - Define the concept of a best response (or best reply)

Best Reply

Lemma

Fix a strategy profile $\left(\sigma_i^*, \sigma_{-i}^*\right)$. *The following statements are equivalent:*

1. $u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*)$ *for all* $\sigma_i \in \Delta\left(S_i\right)$
2. $u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(s_i, \sigma_{-i}^*)$ *for all* $s_i \in S_i$
3. $\sigma_i^*\left(s_i'\right) > 0 \Longrightarrow s_i' \in \arg\max_{s_i} u_i\left(s_i, \sigma_{-i}^*\right)$

Definition

We say σ_i^* is a **best reply** to σ_{-i}^* if one of the above conditions is satisfied. We write $\phi_i\left(\sigma_{-i}^*\right) := \arg\max_{\sigma_i \in \Delta(S_i)} u_i\left(\sigma_i, \sigma_{-i}^*\right)$ as the **best reply correspondence.**

Best Reply

Definition

σ_i^* is a **best reply** to σ_{-i}^* if $\sigma_i^* \in \phi_i\left(\sigma_{-i}^*\right)$. A **Nash equilibrium** is a strategy profile $(\sigma_1^*, ..., \sigma_n^*)$ such that σ_i^* is a best reply to σ_{-i}^*, i.e., $\sigma_i^* \in \phi_i\left(\sigma_{-i}^*\right)$, for any i.

Notes

- Effectively we have been figuring out the best response to find pure strategy Nash Equlibria
- If σ_i^* is a best response, all the stratgies s_i' such that $\sigma_i^*(s_i') > 0$ must have the same payoff
- Does a best response always exist?

Examples

Example

Matching Pennies

	L	R
T	$1,-1$	$-1,1$
B	$-1,1$	$1,-1$

$\Delta(S_1) = \Delta(S_2) = [0,1]$, $p_T = \Pr(T)$ and $p_L = \Pr(L)$. What is player 1's best reply to player 2's strategy p_L?

$$\phi_1(p_L) = \begin{cases} \{1\} & \text{if } p_L > \frac{1}{2} \\ [0,1] & \text{if } p_L = \frac{1}{2} \\ \{0\} & \text{if } p_L < \frac{1}{2} \end{cases}$$

Examples

Example

Matching Pennies

	L	R
T	$1, -1$	$-1, 1$
B	$-1, 1$	$1, -1$

$$\Delta(S_1) = \Delta(S_2) = [0,1]\,,\ p_T = \Pr(T) \text{ and } p_L = \Pr(L)\,.$$

$$\phi_1(p_L) = \begin{cases} \{1\} & \text{if } p_L > \frac{1}{2} \\ [0,1] & \text{if } p_L = \frac{1}{2} \\ \{0\} & \text{if } p_L < \frac{1}{2} \end{cases} ;\ \phi_2(p_T) = \begin{cases} \{1\} & \text{if } p_T < \frac{1}{2} \\ [0,1] & \text{if } p_T = \frac{1}{2} \\ \{0\} & \text{if } p_T > \frac{1}{2} \end{cases}$$

Examples

Example

	L	R
T	$2,1$	$0,0$
B	$0,0$	$1,1$

Pure strategy Nash equilibrium (T,L) and (B,R). Mixed strategy Nash equilibrium? $p_T = \Pr(T)$ and $p_L = \Pr(L)$.

$$\phi_1(p_L) = \begin{cases} \{1\} & \text{if } p_L > \frac{1}{3} \\ [0,1] & \text{if } p_L = \frac{1}{3} \\ \{0\} & \text{if } p_L < \frac{1}{3} \end{cases} ; \phi_2(p_T) = \begin{cases} \{1\} & \text{if } p_T > \frac{1}{2} \\ [0,1] & \text{if } p_T = \frac{1}{2} \\ \{0\} & \text{if } p_T < \frac{1}{2} \end{cases}$$

Existence Theorem

- We have shown that allowing for mixed strategies means that we can find NE in games that have no pure strategy NE
- But is it enough to **guarantee** existence?
- Recall that Nash equilibrium is a strategy profile σ^* such that $\sigma_i^* \in \phi_i\left(\sigma_{-i}^*\right)$ for each i.
- Let us define a correspondence

$$\phi : \prod_{j=1}^{n} \Delta\left(S_j\right) \Longrightarrow \prod_{j=1}^{n} \Delta\left(S_j\right)$$

  such that $\phi\left(\sigma\right) := \left(\phi_1\left(\sigma_{-1}\right), ..., \phi_n\left(\sigma_{-n}\right)\right) = \left(\phi_i\left(\sigma_{-i}\right)\right)$.
- Then, σ^* is a Nash equilibrium iff $\sigma^* \in \phi\left(\sigma^*\right)$, i.e., σ^* is a **fixed point** of ϕ.

Fixed point theorems

Theorem

(Brouwer's fixed point theorem) Let X be a non-empty, convex and compact subset of $\mathbb{R}^N$ for some finite N. Suppose $f : X \to X$ is a continuous function. Then f has a fixed point, i.e., there exists $x^* \in X$ such that $x^* = f(x^*)$.

Theorem

(Kakutani's fixed point theorem) Let X be a non-empty, convex and compact subset of $\mathbb{R}^N$ for some finite N and $F : X \rightrightarrows X$. Suppose

1. *$F(x)$ is non-empty and convex for all $x \in X$*
2. *F is upper-hemi-continuous, i.e., F has a closed graph: $\left(x^k, y^k\right) \to (x, y) \in \mathbb{R}^{2N}$ and $y^k \in F\left(x^k\right)$ imply $y \in F(x)$.*

Then F has a fixed point, i.e., there exists $x^* \in X$ such that $x^* \in F(x^*)$.

Existence of Nash Equilibrium

Theorem

Every finite game has a Nash equilibrium.

Nash Existence (1950 PNAS)

1. Recall that $\Delta(S_i) \subset \mathbb{R}^{|S_i|}$. Consider $X := \prod_{i=1}^{n} \Delta(S_i) \subset \mathbb{R}^{\sum_i |S_i|}$, which is non-empty, convex, and compact.
2. Consider the correspondence $\phi : X \rightrightarrows X$

$$\phi(\sigma) := (\phi_1(\sigma_{-1}), ..., \phi_n(\sigma_{-n})) = (\phi_i(\sigma_{-i})).$$

3. By definition, $\phi_i(\sigma_{-i}) = \arg\max_{\sigma_i} u_i(\sigma_i, \sigma_{-i})$. We can show, $\phi(\sigma)$ is convex.
4. Since u_i is continuous (?) in (σ_i, σ_{-i}), $\phi_i(\sigma_{-i}) = \arg\max_{\sigma_i} u_i(\sigma_i, \sigma_{-i})$ is non-empty and upper-hemi-continuous by the theorem of maximum (MWG M.K.6). Thus ϕ is also non-empty and u.h.c.

Kakutani's fixed point theorem implies that ϕ has a fixed point.

Nash (1951 Annals of Mathematics)

1. $X := \prod_{i=1}^{n} \Delta(S_i)$.
2. Define $f : X \to X$ as follows. For each $\sigma \in \prod_{i=1}^{n} \Delta(S_i)$, $\sigma' = f(\sigma)$ is defined such that for each s_i,

$$\sigma_i'(s_i) = \frac{\sigma_i(s_i) + \max\{0, u_i(s_i, \sigma_{-i}) - u_i(\sigma)\}}{1 + \sum_{s_i'} \max\{0, u_i(s_i', \sigma_{-i}) - u_i(\sigma)\}}.$$

3. **Claim**: $\sigma = f(\sigma)$ if and only if σ is a Nash equilibrium.
   1. That a NE has the property is obvious. Can we show that this implies NE?
   2. Since $u_i(\sigma) = \sum_{s_i} u_i(s_i, \sigma_{-i}) \sigma_i(s_i)$, there exists at least one s_i such that $\sigma_i(s_i) > 0$ and $u_i(s_i, \sigma_{-i}) \leq u_i(\sigma)$.
   3. For this s_i, the above equation becomes (setting $\sigma' = \sigma$),

   $$\sigma_i(s_i) = \frac{\sigma_i(s_i)}{1 + \sum_{s_i'} \max\{0, u_i(s_i', \sigma_{-i}) - u_i(\sigma)\}}$$

   hence $u_i(s_i', \sigma_{-i}) - u_i(\sigma) \leq 0$ for all for all s_i'
   4. Thus it must be the case that $u_i(s_i', \sigma_{-i}) = u_i(\sigma)$

Nash (1951 Annals of Mathematics)

- u_i is continuous in σ; $\max\{0, x\}$ is continuous in x: hence $\sigma_i'(s_i)$ is continuous in σ.
- Hence $f(\sigma)$ is a continuous function in σ. Then Brouwer.

More general strategy space

Theorem

(Fan and Glicksberg) Let X be a nonempty compact convex subset of a convex Hausdorff topological vector space. Let $F : X \rightrightarrows X$ be an upper hemi-continuous convex valued correspondence. Then F has a fixed point.

Theorem

Let S_i be a compact (bounded and closed) and convex subset of $\mathbb{R}^N$. Suppose $u_i : S \to \mathbb{R}$ is continuous. Then there exists a Nash equilibrium (in mixed strategies). Suppose, in addition, u_i is quasi-concave. Then there exists a Nash equilibrium in pure strategies. (Proof Skipped)