G5212: Game Theory

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Modelling Dynamics

- Up until now, our games have lacked any sort of **dynamic** aspect
- We have assumed that all players make decisions at the same time
  - Or at least no player knows what the other has done when they make their decision
- This limits the set of issues we can deal with
- So now we will think about extending our analysis to cases in which players move sequentially

An Intuitive Example

- Think of the standard Cournot game
- There are two firms, call them 1 and 2, producing perfectly substitutable products: market demand is $P(Q) = \max\{a - Q, 0\}$, $Q = q_1 + q_2$, $C(q_i) = cq_i$, $0 < c < a$. The two firms choose quantities simultaneously $q_i \in \mathbb{R}_+$
$$u_i(q_1, q_2) = (P(q_1 + q_2) - c)\, q_i.$$
- For convenience, set $c = 0$, $a = 1$
- What is the Nash equilibrium of this game?
- $q_1^* = q_2^* = \frac{1}{3}$, $u_1(q_1^*, q_2^*) = u_2(q_1^*, q_2^*) = \frac{1}{9}$

An Intuitive Example

- Now let's change the game
- Firm 1 chooses output first
- Firm 2 chooses output conditional on what Firm 1 does
  - Stackleberg Competition
- How might you solve this game?
- Backward induction!
  - First figure out what 2 will do conditional on what 1 does
  - Then assume that 1 maximizes profit **taking into account how player 2 will respond to their actions**

An Intuitive Example

- Firm 2's behavior
  - Say firm 1 chooses $\bar{q}_1$
  - Firm 2 maximizes
$$(1 - q_2 - \bar{q}_1)q_2$$
  - Implies $q_2(\bar{q}_1) = \frac{1-\bar{q}_1}{2}$
- Firm 1's payoff is then

$$\begin{aligned} & (1 - q_2(q_1) - q_1)q_1 \\ = & (1 - \frac{1-q_1}{2} - q_1)q_1 \\ = & \frac{1}{2}(1 - q_1)\, q_1 \end{aligned}$$

  - Implies $\bar{q}_1 = \frac{1}{2}$, $\bar{q}_2 = \frac{1}{4}$, $u_1(\bar{q}_1, \bar{q}_2) = \frac{1}{8}$, $u_2(\bar{q}_1, \bar{q}_2) = \frac{1}{16}$

An Intuitive Example

- Why did the results change?
- Firm 1 takes into account that changes in their behavior will also change the behavior of Firm 2
- This affects their behavior
- In this case the game has a **first mover advantage**

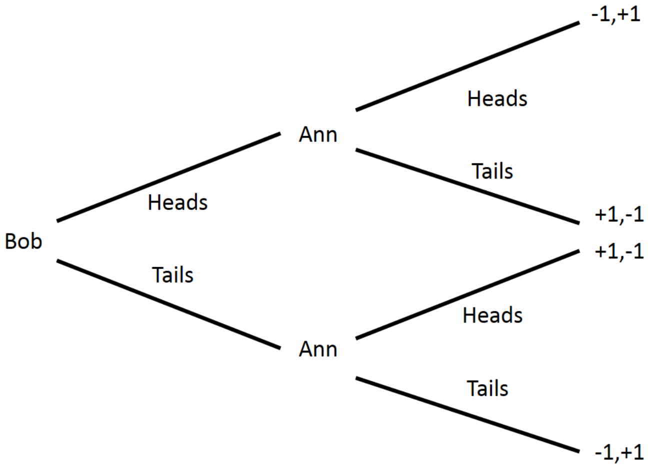
A Second Example

- Let's consider the Matching Pennies game again

		Bob H	Bob T
Anne	H	$+1, -1$	$-1, +1$
	T	$-1, +1$	$+1, -1$

- But now let's assume that Anne moves after Bob
- How could we represent this?
- It can be useful to draw a tree diagram

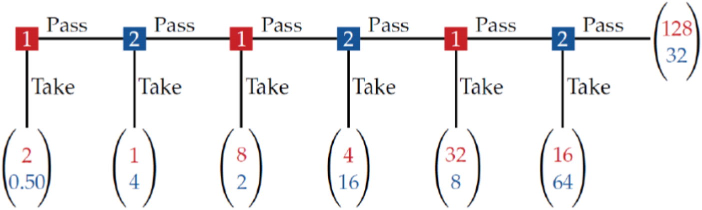
A Second Example



A Second Example

- Again we can solve this through backward induction
  - If Bob plays heads, Anne will play heads
  - If Bob plays tails, Anne will play tails
  - Bob is indifferent between heads and tails
- Two 'sensible' outcomes in pure strategies (H, H) and (T, T)
- Bob gets -1, Anne gets +1
- This is a game with a **second mover advantage**

Sequential Rationality: backward Induction



- While intuitively plausible, backwards induction can have some stark predictions

Formalizing

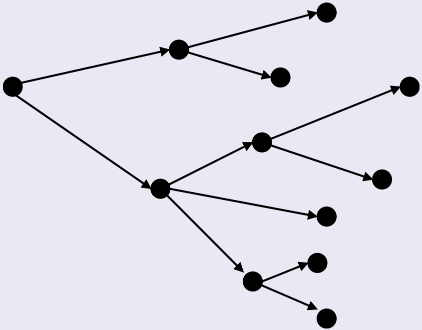
- We will now give a formal definition of a dynamic (or extensive form) game
- Comes from Harold Kuhn (1925-2014)

Defining a Tree

Definition

A **tree** is a set of nodes connected with directed arcs such that

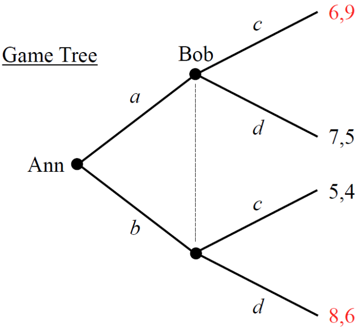
1. There is an initial node;
2. For each other node, there is one incoming arc;
3. Each node can be reached through a unique path.



Kuhn's idea of modelling dynamic games

Definition

A **game** consists of a set of players, a tree, an allocation of each non-terminal node to a player, an information partition, a payoff for each player at each terminal node.



Kuhn's idea of modelling dynamic games

Definition

An **information set** is a collection of nodes such that
1. The same player is to move at each of these nodes;
2. The same moves are available at each of these nodes.

Definition

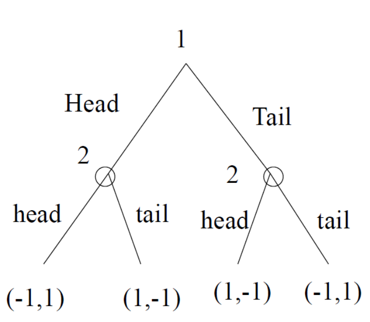
An **information partition** is an allocation of each non-terminal node of the tree to an information set.

- This captures the idea that the player may not always know which node they are at

Kuhn's idea of modelling dynamic games

Example

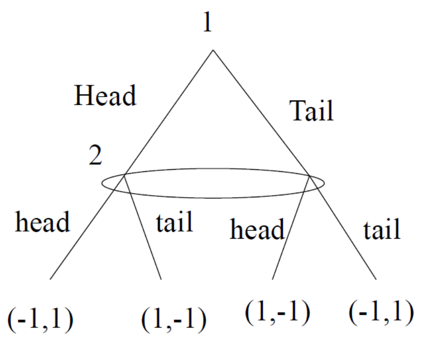
Sequential move matching pennies



Kuhn's idea of modelling dynamic games

Example

Simultaneous move matching pennies



Kuhn's idea of modelling dynamic games

Definition

A (pure) strategy of a player is a complete contingent-plan, determining which action he will take at each information set he is to move (including the information sets that will not be reached according to this strategy).

Definition

A mixed strategy is a distribution over pure strategies.

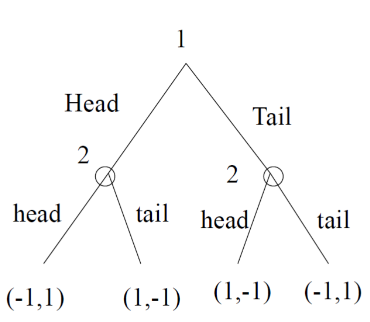
Kuhn's idea of modelling dynamic games

- Note that it is crucial that we now differentiate between **actions** and **strategies**
  - An action is what is chosen by the player at each information set
  - A strategy is a list of actions to be taken at **every** information set at which the player gets to move
- When I ask you to describe a strategy, if you fail to give me an action at each information set, **you are wrong!**

Kuhn's idea of modelling dynamic games

Example

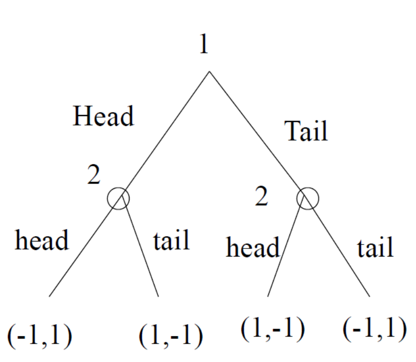
	H	T
H	$-1, 1$	$1, -1$
T	$1, -1$	$-1, 1$



- Two pure strategies for player 1 : H and T.
- Four pure strategies for player 2: s_2 : $\{$Left information set; Right information set$\} \to \{$H,T$\}$.
  - HH; HT; TH; TT

Normal-form representation of extensive-form games

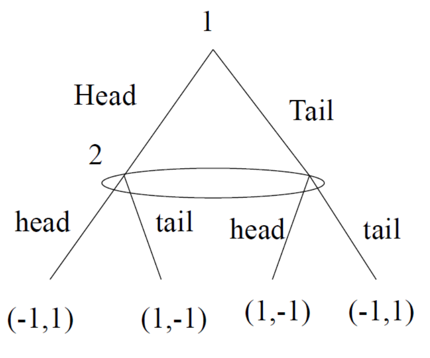
Example



	HH	HT	TH	TT
H	$(-1,1)$	$(-1,1)$	$(1,-1)$	$(1,-1)$
T	$(1,-1)$	$(-1,1)$	$(1,-1)$	$(-1,1)$

Normal-form representation of extensive-form games

Example



	H	T
H	$-1, 1$	$1, -1$
T	$1, -1$	$-1, 1$

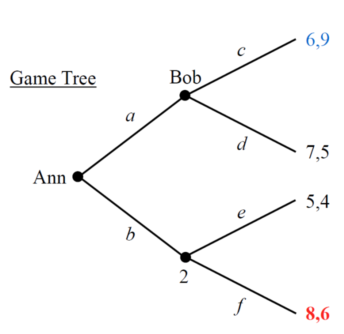
Nash Equilibrium

Definition

A Nash equilibrium of an extensive-form game is defined as Nash equilibrium of its normal-form representation.

Nash Equilibrium

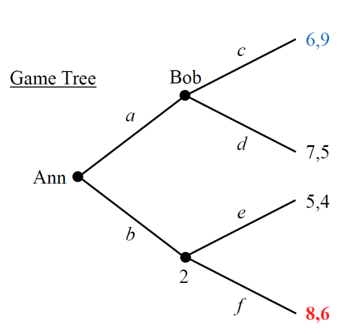
Example



	ce	cf	de	df
a	$(6, 9)$	$(6, 9)$	$(7, 5)$	$(7, 5)$
b	$(5, 4)$	$(8, 6)$	$(5, 4)$	$(8, 6)$

Nash Equilibrium

Example

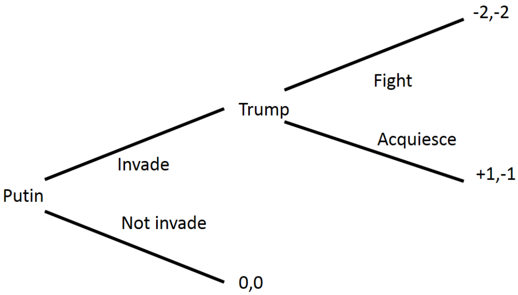


	ce	cf	de	df
a	$(\mathbf{6}, \mathbf{9})$	$(6, 9)$	$(7, 5)$	$(7, 5)$
b	$(5, 4)$	$(\mathbf{8}, \mathbf{6})$	$(5, 4)$	$(\mathbf{8}, \mathbf{6})$

Nash Equilibrium

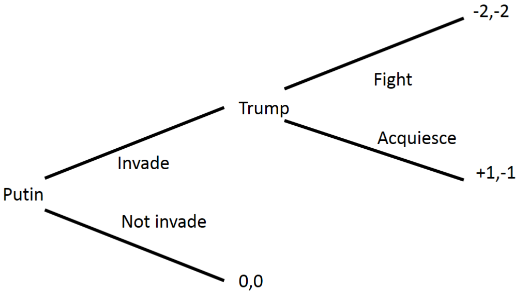
- Three Nash Equilibrium
- Are all of them ‘sensible’?
- Arguably not
  - If Bob ever found himself with the choice between f and e, would choose f
  - But the fact that a is a best response against ce relies centrally on the idea that Bob will play e
- The threat to play e is not **credible**
- The only equilibrium which is consistent with backwards induction is (b, cf)
- Looking at the normal form of the game ignores some crucial dynamic aspects

A Second Example



	Fight	Not Fight
Invade	$(-2,-2)$	$(+1,-1)$
Not Invade	$(0,0)$	$(0,0)$

A Second Example



- Two Nash Equilibrium
- Only one of them is credible
- We need some formal way of capturing this idea of credibility

Backward Induction and Sequential Rationality

- One way of doing so is through the idea of backwards induction.
- We can think of this as solving the game through the assumption of **common knowledge of sequential rationality**

Definition

A player's strategy exhibits sequential rationality if it maximizes his or her expected payoff, conditional on every information set at which he or she has the move. That is, player i's strategy should specify an optimal action at each of player i's information sets, even those that player i does not believe will be reached in the game.

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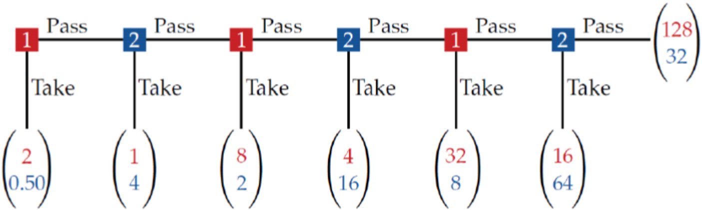
Backward Induction and Sequential Rationality

- Applying common knowledge of sequential rationality justifies solving games by backwards induction
- Players at the last stage will take the optimal action if they ever reach those nodes
- Players at the penultimate stage know that this is how players at the last stage will behave and take the optimal action conditional on this
- and so on

Sequential Rationality: backward Induction



Backward Induction and Sequential Rationality - Comments

- Regardless of past actions, assume all players are playing rationally in future
  - Mistakes in the past do not predict mistakes in the future
- Common knowledge of sequential rationality (CKSR) is different from common knowledge of rationality (CKR).
  - Backward induction is not a result of CKR
  - What if player 1 plays "pass" in the first node?
  - If the "take" action is implied by CKR, then seeing "pass" shakes player 2's conviction of CKR
  - Should player 2 assume player 1 is irrational in future to an extent that player 2 chooses "pass"?
  - If so, player 1, when rational, should intentionally play "pass" in the first node;
  - if not, how should the game continue?
- Arguably, CKSR is quite a strong assumption

Backward Induction and Sequential Rationality - Properties

- Some properties of backwards induction

1. Backward induction always leads to a Nash equilibrium of the normal form of the game, but not all Nash equilibria come about from backwards induction
2. Every complete information finite horizon game can be solved by backward induction
3. The solution is unique for generic games (i.e. ones in which every history leads to a different payoff)