Lecture 9: Bayesian Games

G5212: Game Theory

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 - Players can be uncertain about what the other player will do

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- Payoffs, strategies and players all common knowledge

- So far we have restricted our attention to games in which uncertainty is of a certain kind
 - Players can be uncertain about what the other player will do
 - But they **cannot** be uncertain about the game that they are playing
 - Payoffs, strategies and players all common knowledge
- This is very restrictive
 - Many situations that we can't analyze
 - Particularly situations of asymmetric information
 - This is essentially what the second half of the course will be about

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• Example 1: Signalling

- PhD programs want to recruit people of high ability
- But they cannot observe ability directly
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- PhD programs want to recruit people of high ability
- But they cannot observe ability directly
- Can education be used by high ability candidates to signal that they have high ability?
- Employer uncertain about the quality of the candidate
 - How good the candidate will be if they hire them (own payoff)

- Cost of education to the candidate (other's payoff)
- Information is asymmetric
 - Candidates know their ability, firms do not

- Example 2: Auctions
 - An item is being auctioned on ebay
 - Highest bidder wins the auction and gets the item
 - Each bidder has a different valuation for the item

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• How much should each person bid?

- Example 2: Auctions
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 - Highest bidder wins the auction and gets the item
 - Each bidder has a different valuation for the item
 - How much should each person bid?
 - Each bidder is uncertain about the value of the item to other bidders (other's payoff)
 - (Possibly) the number of bidders
 - Information is asymmetric
 - Each bidder knows their own valuation, not the valuation of everyone else

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- Example 3: Currency Attacks
 - The strength of a currency is uncertain
 - There are a number of speculators deciding whether to attack the currency
 - The number who need to attack to bring the currency down depends on its strength

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- Example 3: Currency Attacks
 - The strength of a currency is uncertain
 - There are a number of speculators deciding whether to attack the currency
 - The number who need to attack to bring the currency down depends on its strength
 - Each player receives a signal about the strength of the currency
 - Each bidder uncertain about the strength of the currency (own and others payoff)

- Also uncertain about what signal others have got (information of others)
- Information is asymmetric
 - Each bidder knows their own signal

• Cournot Competition with unknown types

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- Probability of each type of firm

$$p(c_1^H, c_2^H) = p(c_1^L, c_2^L) = \frac{1}{2}\alpha$$

$$p(c_1^H, c_2^L) = p(c_1^L, c_2^H) = \frac{1}{2}(1 - \alpha)$$

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• Payoffs for cost $c_i \in \left\{c_i^H, c_i^L\right\}$ and $q_i \in \left\{q_i^H, q_i^L\right\}$ $q_i \left(P(Q) - c_i\right)$ Lecture 9: Bayesian Games

A Worked Example

• How do we solve this problem?



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- Let's think about the behavior of player 1
- Fix a strategy for player 2: $\{q_2^H, q_2^L\}$
- If player 1 has costs c^H what is the probability that player 2 is H?

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- If player 1 has costs c^H what is the probability that player 2 is H?
- Use Bayes' rule

$$P(c_{2}^{H}|c_{1}^{H}) = \frac{P(c_{2}^{H} \cap c_{1}^{H})}{P(c_{1}^{H})} \\ = \frac{\frac{1}{2}\alpha}{\frac{1}{2}\alpha + \frac{1}{2}(1-\alpha)} = \alpha$$

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Lecture 9: Bayesian Games

A Worked Example

• If player 1 is of type c^H their optimization problem is therefore

$$\max_{q} \left[\alpha (1 - (q_2^H + q) - c_1^H)q + (1 - \alpha)(1 - (q_2^L + q) - c_1^H)q \right].$$

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$$\max_{q} \left[\alpha (1 - (q_2^H + q) - c_1^H)q + (1 - \alpha)(1 - (q_2^L + q) - c_1^H)q \right].$$

• FOC:

$$\alpha \left(1 - c_1^H - q_2^H - 2q \right) + (1 - \alpha) \left(1 - c_1^H - q_2^L - 2q \right) = 0$$

or

$$q = \frac{1 - c_1^H - \alpha q_2^H - (1 - \alpha) q_2^L}{2}$$

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Cournot competition with private cost

• We can repeat the exercise for the case in which player 1 has c^L

$$P(c_2^H | c_1^L) = \frac{P(c_2^H \cap c_1^L)}{P(c_1^L)} \\ = \frac{\frac{1}{2}(1-\alpha)}{\frac{1}{2}\alpha + \frac{1}{2}(1-\alpha)} = (1-\alpha)$$

• So they must maximize

$$\max_{q} \left[(1-\alpha)(1-(q_{2}^{H}+q)-c_{1}^{L})q + \alpha(1-(q_{2}^{L}+q)-c_{1}^{L})q \right].$$

• Giving first order conditions

$$q = \frac{1 - c_1^L - (1 - \alpha)q_2^H - \alpha q_2^L}{2}$$

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Cournot competition with private cost

• Repeating the exercise for Player 2 gives 4 equations and 4 unknowns

$$\begin{split} q_1^H &= \frac{1 - c_1^H - \alpha q_2^H - (1 - \alpha) q_2^L}{2} \\ q_1^L &= \frac{1 - c_1^L - (1 - \alpha) q_2^H - \alpha q_2^L}{2} \\ q_2^H &= \frac{1 - c_2^H - \alpha q_1^H - (1 - \alpha) q_1^L}{2} \\ q_2^L &= \frac{1 - c_2^L - (1 - \alpha) q_1^H - \alpha q_1^L}{2} \end{split}$$

• Which can be solved to get the solution to this problem

- Let's now try to formalize what we just did
- First the set up
 - N: Set of players (firms)
 - $T = \{T_n\}_{n \in N}$: Types space for each player (possible cost of each firm)
 - $P \in \Delta(T)$: prior probabilities over types (α)
 - $S = \{S_n\}_{n \in \mathbb{N}}$: Strategy space for each player (output levels)

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• $u: S \times T \to \mathbb{R}^N$: Payoffs function for each player (profits)

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 - $S = \{S_n\}_{n \in \mathbb{N}}$: Strategy space for each player (output levels)
 - $u: S \times T \to \mathbb{R}^N$: Payoffs function for each player (profits)
- Notes
 - $\bullet~P$ the same for each player no agreeing to disagree
 - S_n doesn't depend on t_n
 - If we restrict u_n to only depend on t_n this is the case of **independent values**

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- Solution
- Every play picked a (mixed) strategy $\sigma_n^* : T_n \to \Delta(S_i)$ (production levels conditional on type)
- In order to maximize

$$\mathbb{E}\left[u_{n}\left(\sigma_{n},\sigma_{-n}^{*}\left(t_{-n}\right),t\right)|t_{n}\right]$$

• Where

$$\mathbb{E}\left[u_{n}\left(\sigma_{n},\sigma_{-n}^{*}\left(t_{-n}\right),t\right)|t_{n}\right]$$

$$=\sum_{t_{-n}\in T_{-n}}u_{n}\left(\sigma_{n},\sigma_{-n}^{*}\left(t_{-n}\right),t\right)p\left(t_{-n}|t_{n}\right).$$

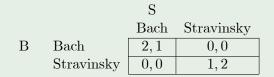
$$p\left(t_{-n}|t_{n}\right)=\frac{p\left(t_{-n},t_{n}\right)}{p(t_{n})}$$

• This is the **Bayesian Nash Equilibrium** of the game

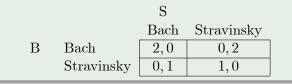
Example

Consider a modified BOS game in which the type of the Splayer is unknown

With prob 1/2 the payoff matrix is 1/2 (friendly)



With prob 1/2 the payoff matrix is (unfriendly)



Lecture 9: Bayesian Games



• What is the best response of the S player?



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- BR(B|Friendly) = B
- BR(S|Friendly) = S
- BR(B|Unfriendly) = S
- BR(S|Unfriendly) = B

• What is the best response of the *B* player?



- What is the best response of the *B* player?
- Remember, the B player best responds to a ${\bf strategy}$ of the S player
 - Action conditional on type
 - $\bullet\,$ e.g. BS : Bach if friendly, Stravinsky if unfriendly

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• Payoff for playing B against BS is $\frac{1}{2}2 + \frac{1}{2}0$

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- Payoff for playing B against BS is $\frac{1}{2}2 + \frac{1}{2}0$
- \circ So
 - BR(BB) = B
 - BR(BS) = B
 - BR(SB) = B
 - BR(SS) = S



• Can it be part of a NE for Stravinsky to play S?

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• Can it be part of a NE for Stravinsky to play S?

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- No
- $\bullet\,$ Best response to S is SB
- Best response to SB is B

- Can it be part of a NE for Stravinsky to play S?
 - No
 - Best response to S is SB
 - Best response to SB is B
- Can it be part of a NE for Stravinsky to play B?

- Can it be part of a NE for Stravinsky to play S?
 - No
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 - Best response to SB is B
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- Yes
- Best response to B is BS
- Best response to BS is B