G5212: Game Theory

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Adverse Selection

- Last lecture we discussed a specific model of adverse selection
  - Two types for the agent
- This gave us a lot of intuition
  - Indeed, there were many lessons that will turn out to be general

The Second Best Contract

- Summarizing the second best contract
  - High type gets efficient allocation
  - Every type is indifferent between their contract and that of the type immediately below
  - All types but the lowest get positive surplus (informational rent)
  - All types but the highest get less than efficient allocation
  - Lowest type gets zero surplus

Adverse Selection

- However, it was also rather specific
- Today we will solve a more general model
  - Agent has a type $\theta \in [\theta_1, \theta_2]$ - continuous type space
  - Distributed according to continuous pdf f with cdf F
  - Principal and agent will exchange a bundle of goods q for payment t
  - Utility of principal
$$W(q,t)$$
  - Utility of agent of type θ
$$U(q,t,\theta) = U(q,\theta) - t$$
- We will assume model is parameterized in such a way that all types are sold to

Optimality

- As in the previous model we can assume that the principal offers the agent a set of contracts

$$q(\theta),\ t(\theta)$$

  Which is the bundle of goods and price designed for the agent of type θ
- Two types of constraints must hold:
- Individual rationality: Each type of agent must prefer their contract to the outside option (which we normalize to 0)

$$U(q(\theta), t(\theta), \theta) \geq 0$$

- Incentive Compatibility: Each agent must prefer their own contract to any other

$$U(q(\theta), t(\theta), \theta) \geq U(q(\hat{\theta}), t(\hat{\theta}), \theta)$$

Incentive Compatibility

- We will start by analyzing the IC constraints
- Define

$$V(\theta, \hat{\theta}) = U(q(\hat{\theta}), t(\hat{\theta}), \theta)$$

  Utility of an agent of type θ who announces $\hat{\theta}$

- IC constraint says, for all $\theta \in [\theta_1, \theta_2]$

$$V(\theta, \theta) \geq V(\theta, \hat{\theta})$$

First and Second Order Conditions

- This means that $V(\theta, \theta)$ must be an optimum of $V(\theta, \hat{\theta})$
- We will assume that the mechanism is differentiable
- If so, necessary conditions are, for all θ
    - First order conditions

$$\left.\frac{\partial V}{\partial \hat{\theta}}\right|_{\hat{\theta}=\theta} (\theta, \hat{\theta}) = 0$$

    - Second order conditions

$$\left.\frac{\partial^2 V}{\partial \hat{\theta}^2}\right|_{\hat{\theta}=\theta} (\theta, \hat{\theta}) \leq 0$$

First Order Condition

$$\left.\frac{\partial V}{\partial \hat{\theta}}\right|_{\hat{\theta}=\theta}(\theta, \hat{\theta}) = 0$$

- This implies

$$\left.\frac{\partial}{\partial \hat{\theta}}\right|_{\hat{\theta}=\theta}\left(U(q(\hat{\theta}), \theta) - t(\hat{\theta})\right) = 0$$

- or

$$t'(\hat{\theta}) = \frac{\partial U(q(\hat{\theta}), \theta)}{\partial q} q'\left(\hat{\theta}\right)$$

- For $\hat{\theta} = \theta$

Second Order Condition

$$\left.\frac{\partial^2 V}{\partial \hat{\theta}^2}\right|_{\hat{\theta}=\theta} (\theta, \hat{\theta}) \leq 0$$

- This implies

$$\left.\frac{\partial^2}{\partial \hat{\theta}^2}\right|_{\hat{\theta}=\theta} \left[U(q(\hat{\theta}), \theta) - t(\hat{\theta})\right] \leq 0$$

- or

$$t''(\hat{\theta}) \geq \frac{\partial U(q(\hat{\theta}), \theta)}{\partial q} q''(\hat{\theta}) + \frac{\partial^2 U(q(\hat{\theta}), \theta)}{\partial q^2} \left(q'(\hat{\theta})\right)^2$$

- For $\hat{\theta} = \theta$

Second Order Condition

- Notice that the FOC must hold for every θ
- So differentiate WRT θ, taking into account that $\hat{\theta}$ must move 1-1 with θ

$$t'(\hat{\theta}) = \frac{\partial U(q(\hat{\theta}), \theta)}{\partial q} q'\left(\hat{\theta}\right)$$

- implies

$$\begin{aligned} t''(\hat{\theta}) &= \frac{\partial U(q(\hat{\theta}), \theta)}{\partial q} q''\left(\hat{\theta}\right) \\ &+ \frac{\partial^2 U(q(\hat{\theta}), \theta)}{\partial q^2}\left(q'(\hat{\theta})\right)^2 \\ &+ \frac{\partial^2 U(q(\hat{\theta}), \theta)}{\partial q \partial \theta} q'(\hat{\theta}) \end{aligned}$$

Second Order Condition

- Substituting back in to the SOC gives

$$\left.\frac{\partial^2 U(q(\hat{\theta}),\theta)}{\partial q \partial \theta}\right|_{\theta=\hat{\theta}} q'(\hat{\theta}) \geq 0$$

- This gives us two conditions stemming from the FOC and SOC of the IC constraints for $\hat{\theta} = \theta$
  - IC1

$$t'(\hat{\theta}) = \frac{\partial U(q(\hat{\theta}),\theta)}{\partial q} q'\left(\hat{\theta}\right)$$

  - IC2

$$\frac{\partial^2 U(q(\hat{\theta}),\theta)}{\partial q \partial \theta} q'(\hat{\theta}) \geq 0$$

Second Order Condition

- Notice that IC2 depends crucially on

$$\frac{\partial^2 U(q(\hat{\theta}), \theta)}{\partial q \partial \theta}$$

- Which is a property of the utility function
- Life becomes much easier if we assume that this has the same sign everywhere
  - e.g. it is strictly positive

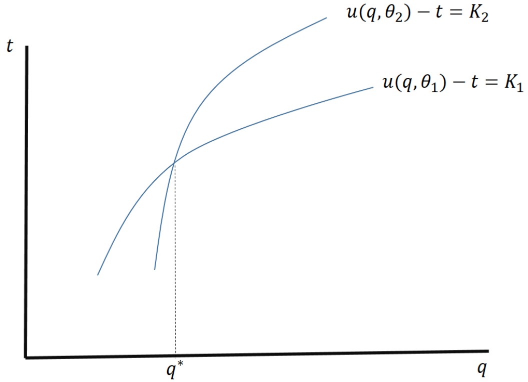
Second Order Condition

- This is the Spence/Mirrlees condition
  - Also known as the single crossing condition
- It has economic content:
- Means (for example) that

$$\frac{\partial U(q,\theta)}{\partial q}$$

  is always increasing in type
- The marginal utility of an increase in q is always higher for a higher type
  - or, in other words, the willingness to pay for an increase in q is always higher for higher types

Single Crossing



- For $\theta_2 > \theta_1$

Single Crossing

- Why does Spence-Mirrlees imply single crossing?
- Equation for the type i indifference curve

$$t(q,\theta) = U(q,\theta_i) - K$$

- Slope is therefore given by

$$\frac{\partial U(q,\theta)}{\partial q}$$

- Pick the q^* where the two indifference curves cross
  - For any $q > q^*$

$$\begin{aligned} t(q,\theta_2) - t(q,\theta_1) &= \int_{q^*}^{q} \frac{\partial U(q,\theta_2)}{\partial q} d(q) - \int_{q^*}^{q} \frac{\partial U(q,\theta_1)}{\partial q} d(q) \\ &\quad + t(q^*,\theta_2) - t(q^*,\theta_1) \\ &= \int_{q^*}^{q} \frac{\partial U(q,\theta_2)}{\partial q} - \frac{\partial U(q,\theta_1)}{\partial q} d(q) > 0 \end{aligned}$$

Single Crossing

- Spence-Mirrlees condition is sometimes called a Sorting condition

Theorem

$q : [\theta_1, \theta_2] \rightarrow \mathbb{R}$ *belongs to a direct truthful mechanism* ***if*** *and only if it is non-decreasing*

- Meaning: for any non-decreasing q we can find a t function that makes it part of a truth telling mechanism

Single Crossing

- Proof
  - Start with

$$\frac{\partial V}{\partial \hat{\theta}}(\theta, \hat{\theta}) = \frac{\partial U(q(\hat{\theta}), \theta)}{\partial q} q'\left(\hat{\theta}\right) - t'(\hat{\theta})$$

  - At the optimum for type $\hat{\theta}$ we know from IC1

$$t'(\hat{\theta}) = \frac{\partial U(q(\hat{\theta}), \hat{\theta})}{\partial q} q'\left(\hat{\theta}\right)$$

  - And so

$$\frac{\partial V}{\partial \hat{\theta}}(\theta, \hat{\theta}) = \left[\frac{\partial U(q(\hat{\theta}), \theta)}{\partial q} - \frac{\partial U(q(\hat{\theta}), \hat{\theta})}{\partial q}\right] q'\left(\hat{\theta}\right)$$

Single Crossing

- Stare at

$$\frac{\partial V}{\partial \hat{\theta}}(\theta, \hat{\theta}) = \left[\frac{\partial U(q(\hat{\theta}), \theta)}{\partial q} - \frac{\partial U(q(\hat{\theta}), \hat{\theta})}{\partial q} \right] q'\left(\hat{\theta}\right)$$

- And recall that

$$f(x) = f(\hat{x}) + f'(x^*)(x - \hat{x})$$

  for $x^* \in (x, \hat{x})$
- And so

$$\frac{\partial V}{\partial \hat{\theta}}(\theta, \hat{\theta}) = \left[\frac{\partial^2 U(q(\hat{\theta}), \theta^*)}{\partial q \partial \theta} \left(\theta - \hat{\theta}\right) \right] q'\left(\hat{\theta}\right)$$

Single Crossing

- Given the Spence-Mirrlees condition, this will have the same sign as $\left(\theta - \hat{\theta}\right)$ as long as $q\left(\hat{\theta}\right)$ is non-decreasing
- Implies $V(\theta, \hat{\theta})$
  - is increasing in $\hat{\theta}$ for $\hat{\theta} < \theta$
  - is decreasing in $\hat{\theta}$ for $\hat{\theta} > \theta$
- Global maximizer at $\hat{\theta} = \theta$

Single Crossing

- This is extremely useful
  - Started off with a huge number of global IC conditions
- Turns out we can concentrate on a couple of local conditions
- IC2: implies that q is non-decreasing
- IC1: gives us t from

$$t'(\hat{\theta}) = \frac{\partial U(q(\hat{\theta}), \theta)}{\partial q} q'\left(\hat{\theta}\right)$$

- Problem boils down to picking non-decreasing function q (and associated t) that maximizes profit

Solving the Model

- Assume that the principal gets profit of the following type

$$t - C(q)$$

- And that agents satisfy

$$\frac{\partial U}{\partial \theta}(q, \theta) > 0$$

$$\frac{\partial^2 U(q(\hat{\theta}), \theta)}{\partial q \partial \theta} > 0$$

Solving the Model

- Define

$$v(\theta) = V(\theta, \theta) = u(q(\theta), \theta) - t(\theta)$$

As the information rent that agent of type

- The derivative of this is

$$\begin{aligned} v'(\theta) &= \frac{\partial u(q(\theta), \theta)}{\partial q} q'(\theta) - t'(\theta) + \frac{\partial u(q(\theta), \theta)}{\partial \theta} \\ &= \frac{\partial u(q(\theta), \theta)}{\partial \theta} \end{aligned}$$

  - (note that this is the derivative of u wrt 2nd argument)
- By IC 1
- By assumption, this is greater than 0
  - Higher types get higher information rents

Solving the Model

- Assume (for now) that IR constraints are the same for all types

$$v(\theta) \geq 0$$

- Or, by the above result

$$v(\theta_1) = 0$$

- IR of the lowest type is the only one that binds

Fundamental Theorem of Calculus

- Generally we know

$$f(x) = \int_{x_1}^{x} f'(y)d(y) + f(x_1)$$

- So we can rewrite

$$\begin{aligned} v(\theta) &= \int_{\theta_1}^{\theta} v'(\tau)d\tau + v(\theta_1) \\ &= \int_{\theta_1}^{\theta} \frac{\partial u(q(\tau),\tau)}{\partial \theta} d(\tau) \end{aligned}$$

- And pin down transfers as

$$\begin{aligned} t(\theta) &= u(q(\theta),\theta) - v(\theta) \\ &= u(q(\theta),\theta) - \int_{\theta_1}^{\theta} \frac{\partial u(q(\tau),\tau)}{\partial \theta} d(\tau) \end{aligned}$$

The Principal's Objective

- The Principal wants to maximize expected profit

$$\int_{\theta_1}^{\theta_2} (t(\theta) - C(q(\theta))\, f(\theta) d\theta$$

- Substituting in gives

$$\int_{\theta_1}^{\theta_2} \left(u(q(\theta), \theta) - \int_{\theta_1}^{\theta} \frac{\partial u(q(\tau), \tau)}{\partial \theta} d(\tau) - C(q(\theta)) \right) f(\theta) d\theta$$

- Leaving us with a double integral (eek!)

Integration by Parts

- We can make more progress by integrating by parts

$$
\begin{aligned}
G(\theta) &= \int_{\theta_1}^{\theta} \frac{\partial u(q(\tau),\tau)}{\partial \theta} d(\tau) \\
&\Rightarrow \int_{\theta_1}^{\theta_2} G(\theta) f(\theta) d(\theta) \\
&= [G(\theta)F(\theta)]_{\theta_1}^{\theta_2} - \int_{\theta_1}^{\theta_2} G'(\theta)F(\theta)d(\theta) \\
&= G(\theta_2) - \int_{\theta_1}^{\theta_2} G'(\theta)F(\theta)d(\theta) \\
&= \int_{\theta_1}^{\theta_2} \frac{\partial u(q(\theta),\theta)}{\partial \theta} d(\theta) - \int_{\theta_1}^{\theta_2} \frac{\partial u(q(\theta),\theta)}{\partial \theta} F(\theta) d(\theta) \\
&= \int_{\theta_1}^{\theta_2} \frac{\partial u(q(\theta),\theta)}{\partial \theta} (1 - F(\theta)) d(\theta)
\end{aligned}
$$

Rewriting the Principal's Problem

- And so we can write the Principal's objective as

$$\int_{\theta_1}^{\theta_2}\left(u(q(\theta),\theta)-\int_{\theta_1}^{\theta}\frac{\partial u(q(\tau),\tau)}{\partial\theta}d(\tau)-C(q(\theta))\right)f(\theta)d\theta$$

$$=\int_{\theta_1}^{\theta_2}H(q(\theta),\theta)f(\theta)d\theta$$

- Where

$$H(q,\theta)=u(q,\theta)-C(q)-\frac{\partial u(q,\theta)}{\partial\theta}\frac{1}{h(\theta)}$$

and

$$h(\theta)=\frac{f(\theta)}{1-F(\theta)}$$

- is the hazard rate
  - Prob of being type θ given at least type θ

Rewriting the Principal's Problem

$$H(q,\theta) = u(q,\theta) - c(q) - \frac{\partial u(q,\theta)}{\partial \theta}\frac{1}{h(\theta)}$$

- This is the virtual surplus
  - First part is total surplus
  - Second part is the distortion necessary for the IC constraint to hold

Solving the Problem

$$H(q,\theta) = u(q,\theta) - C(q) - \frac{\partial u(q,\theta)}{\partial \theta}\frac{1}{h(\theta)}$$

- How do we proceed?
- Well, we can start by picking q to maximize H for every θ.
- Will this necessarily be the solution?
- No, because the second order IC constrains may not hold
- i.e. it does not guarantee

$$q'(\theta) \geq 0$$

- Cross our fingers and assume it works

First Order Conditions (again!)

$$
\begin{aligned}
H_q(q,\theta) &= 0 \\
\Rightarrow \quad & u_q(q,\theta) - C'(q) - \frac{\partial^2 u(q,\theta)}{\partial\theta\partial q}\frac{1}{h(\theta)}
\end{aligned}
$$

- Define $q^*(\theta)$ as the solution to this equation
- If this function is non-decreasing then we are done!
  - Perfect separation
  - Perfect revelation
- Can guarantee this with auxiliary assumptions, e.g
  - $u(q,\theta) = q\theta$
  - C convex
  - Hazard rate non-decreasing

Bunching Solutions

- If $q^*(\theta)$ is decreasing for some portion, it cannot be the solution
- We can divide the optimal function into two types of region on the type space
  - q is increasing $\Rightarrow H_q(q,\theta) = 0$ and $q(\theta) = q^*(\theta)$
  - $q(\theta)$ is constant