G5212: Game Theory

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Moral Hazard - Extensions

- Now we have introduced the concept of moral hazard in a simple 2x2 case we will move on to some more complex cases
- Unfortunately, the analysis will be less satisfying
  - Need to make further assumptions to say much that is concrete
- We will look at three cases
  - The discrete case (standard model)
  - The continuous case
  - Liquidity constraints

The Standard Model

- We will now analyze the 'standard' version of the model
  - Which we will do with discrete actions and states
- n possible actions $a_1, ...a_n$
- m possible outcomes $x_1, ...x_m$
- Technology: probability of each outcome given each action

$$p_i = \begin{pmatrix} p_{i1} \\ \vdots \\ p_{im} \end{pmatrix}$$

where p_{ij} is the probability of outcome j given action i

The Standard Model

- Contract takes the form of wages w_j for each outcome j
  - This is the only thing that the principal can condition wages on
- Means that if outcome x_j occurs then
  - Principal receives $x_j - w_j$
  - Agent receives w_j
- Note that this means that the information that the principal receives is only in the form of their 'income'
  - Will come back to this later
- Assume quasi linear utility for the agent

$$u(w) - a$$

- u increasing and concave

The Agent's Problem

- Given a wage schedule w what will the agent do?
- If they work, they will choose action a_i to maximize

$$\sum_{j=1}^{m} p_{ij} u(w_j) - a_i$$

- Having determined this action, will have to decide whether to work or not - i.e. choose

$$\max \left\{ \sum_{j=1}^{m} p_{ij} u(w_j) - a_i, \bar{u} \right\}$$

The Principal's Problem

- We can think of the principal's problem in two stages
    1. Determine the lowest cost way of implementing any action a_i
    2. Choose the a_i that maximizes expected profit given costs
- We will concentrate on stage 1

The Principal's Problem

- Let's assume that the agent wants to implement an action a_i
- What constraints do they have to worry about?
- $n-1$ IC constraints

$$\sum_{j=1}^{m} p_{ij} u(w_j) - a_i \geq \sum_{j=1}^{m} p_{kj} u(w_j) - a_k$$

  for every $k \neq i$
- One IR constraint

$$\sum_{j=1}^{m} p_{ij} u(w_j) - a_i \geq \bar{u}$$

The Principal's Problem

- We can therefore set the principal's problem up as a Lagrangian

$$
\begin{aligned}
L &= \sum_{j=1}^{m} p_{ij}(x_j - w_j) \\
&\quad + \sum_{k \neq i} \lambda_k \left(\sum_{j=1}^{m} (p_{ij} - p_{kj})\, u(w_j) - (a_i - a_k) \right) \\
&\quad + \mu \left(\sum_{j=1}^{m} p_{ij} u(w_j) - a_i - \bar{u} \right)
\end{aligned}
$$

- With $\lambda_k, \mu \geq 0$

The Principal's Problem

- First order conditions give

$$\frac{1}{u'(w_j)} = \sum_{k \neq i} \lambda_k \left(1 - \frac{p_{kj}}{p_{ij}}\right) + \mu$$

- What does this mean?
  - Depends on which constraints are binding
  - First best solution, no IC
  - $\lambda_k = 0 \ \forall \ k$
  - Implies

$$\frac{1}{u'(w_j)} = \mu$$

  - Marginal utility constant across j
  - Wages are constant across j

The Principal's Problem

- More generally, some IC constraint will bind
  - Say on action k
  - This means that at the solution the agent will be indifferent between action i and action k

$$\sum_{j=1}^{m} p_{ij} u(w_j) - a_i = \sum_{j=1}^{m} p_{kj} u(w_j) - a_k$$

  - And $\lambda_k > 0$

The Principal's Problem

- What effect will that have on the wage after outcome j?

$$\frac{1}{u'(w_j)} = \sum_{k \neq i} \lambda_k \left(1 - \frac{p_{kj}}{p_{ij}}\right) + \mu$$

- Depends on whether $\frac{p_{kj}}{p_{ij}}$ is greater or less than 1
  - If it is less than 1 then $\lambda_k \left(1 - \frac{p_{kj}}{p_{ij}}\right)$ positive, so constraint raises $\frac{1}{u'(w_j)}$ and so w_j
  - If it is greater than 1 then $\lambda_k \left(1 - \frac{p_{kj}}{p_{ij}}\right)$ negative, so constraint lowers $\frac{1}{u'(w_j)}$ and so w_j

The Principal's Problem

- So what is $\frac{p_{kj}}{p_{ij}}$?
- It is the relative likelihood of outcome j if action k is taken to if action i is taken
- In fact, the maximum likelihood estimator of the action a conditional observing outcome j is the action h such that

$$\frac{p_{kj}}{p_{hj}} \leq 1 \text{ for all } k$$

- So if the principal
  - is worried about the agent playing k
  - for some outcome j $\frac{p_{kj}}{p_{ij}} > 1$
  - This means that lowering the wage in that state punishes action k more than i
  - Hence will lower the wage in that state
- Change in wage from first best is the sum across all such effects

The Principal's Problem

- Assume
  - $x_j > x_k$ for $j > k$
  - $a_j > a_k$ for $j > k$
  - What can we learn about the optimal wage contract?
- Unfortunately without additional assumptions, not very much
- We would like to be able to say (for example) that wages and profits are increasing in outcomes
  - This was true in the simple 2 action 2 outcome case
- Unfortunately this is not generally true

The Principal's Problem

- Grossman and Hart [1983] show that in generally all we can say is
  - w_j cannot be uniformly decreasing
  - $(x_j - w_j)$ cannot be uniformly decreasing
  - There is a $j > k$ such that $w_j > w_k$ and $x_j - w_j \geq x_k - w_k$
- Proof beyond the scope of this course
- But it means that the solution is only guaranteed to be well behaved in the case of 2 states and 2 actions
- Where we can write the solution as

$$\begin{aligned} w_1 &= w \\ w_2 &= w + s(x_2 - x_1) \end{aligned}$$

- for $0 < s \leq 1$

The Principal's Problem

- To say more we need to put more structure on p
- Return to the solution of the first order conditions

$$\frac{1}{u'(w_j)} = \sum_{k \neq i} \lambda_k \left(1 - \frac{p_{kj}}{p_{ij}}\right) + \mu$$

- If we want the solution to have the 'natural' assumption that wages are increasing in output, what do we need?
- Well, we need the RHS to be increasing in j
- How can this be achieved?

The Monotone Likelihood Ratio Condition

- We are going to need two ingredients
- The first is that we require that better actions make better outcomes more likely
- This is the monotone likelihood ratio condition
- For $k < i$ and $l < j$

$$\frac{p_{ij}}{p_{il}} \geq \frac{p_{kj}}{p_{kl}}$$

  - Switching to the better action i increases the probability of the better outcome j relative to the worse outcome l

The Monotone Likelihood Ratio Condition

Example

Say that

$$p_k = \begin{pmatrix} 0.5 \\ 0.3 \\ 0.2 \end{pmatrix} \quad p_i = \begin{pmatrix} 0.2 \\ 0.3 \\ 0.5 \end{pmatrix}$$

The the MLRC situation is satisfied

$$\begin{aligned} \frac{p_{k3}}{p_{k2}} &= \frac{2}{3} \; \frac{p_{i3}}{p_{i2}} = \frac{5}{3} \\ \frac{p_{k3}}{p_{k1}} &= \frac{2}{5} \; \frac{p_{i3}}{p_{i2}} = \frac{5}{2} \\ \frac{p_{k2}}{p_{k1}} &= \frac{3}{5} \; \frac{p_{i2}}{p_{i1}} = \frac{3}{2} \end{aligned}$$

The Monotone Likelihood Ratio Condition

Example

Say that

$$p_k = \begin{pmatrix} 0.2 \\ 0.4 \\ 0.3 \end{pmatrix} \quad p_i = \begin{pmatrix} 0.3 \\ 0.3 \\ 0.4 \end{pmatrix}$$

The the MLRC situation is not satisfied

$$\begin{aligned} \frac{p_{k3}}{p_{k2}} &= \frac{3}{4} \; \frac{p_{i3}}{p_{i2}} = \frac{4}{3} \\ \frac{p_{k3}}{p_{k1}} &= \frac{3}{2} \; \frac{p_{i3}}{p_{i2}} = \frac{4}{3} \\ \frac{p_{k2}}{p_{k1}} &= \frac{4}{2} \; \frac{p_{i2}}{p_{i1}} = \frac{3}{3} \end{aligned}$$

The Monotone Likelihood Ratio Condition

- Is this enough to guarantee that

$$\frac{1}{u'(w_j)} = \sum_{k \neq i} \lambda_k \left(1 - \frac{p_{kj}}{p_{ij}}\right) + \mu$$

  is increasing in j?
- Not yet
- We know that

$$\left(1 - \frac{p_{kj}}{p_{ij}}\right)$$

  is increasing in j if $k < i$, but decreasing otherwise

The Convexity of The Distribution Function Assumption

- This brings us to our second ingredient
- If we can makes sure that only the constraints on **lower** effort levels are binding
- This means that $\lambda_k = 0$ for $k > i$
- Guarantees that w_j is increasing in output

The Convexity of The Distribution Function Assumption

- Under what circumstances can we guarantee this?
- One is if we are targeting the highest possible effort level
- A second is **convexity of the distribution function**
  - Sort of like a decreasing returns assumption

Definition

We say a distribution function p is convex if, for $i < j < k$ and λ such that

$$a_j = \lambda a_i + (1 - \lambda) a_k$$

we have

$$P_{jl} \leq \lambda P_{il} + (1 - \lambda) P_{kl}$$

where P_j is the CDF of p_j

Solution to the Principal's Problem

- We are now in a position to fully characterize the solution to the principal's problem under MLR and CDFC
- Assume we are targeting level a_i
- First note that there must be a binding constraint for some $l < i$
    - Assume not
    - $\lambda_l = 0$ for all $l < i$
    - This means solution would be the same as to a problem which excluded these actions
    - In which case a_i would be the lowest effort level
    - Means wages would be constant
    - But this cannot induce effort a_i in the original problem

Solution to the Principal's Problem

- Second, consider the solution to a problem in which we remove all acts higher that i
  - a_i is the highest act, so we know that the wage is increasing in j (assuming MLRC)
- Claim: This wage schedule is still IC for the original problem (assuming CDCF)
- Assume not, and there is some $k > i$ such that

$$\sum_{j=1}^{m} p_{kj} u(w_j) - a_k > \sum_{j=1}^{m} p_{ij} u(w_j) - a_i$$

- Let $l < i$ index the act with the binding constraint

$$\sum_{j=1}^{m} p_{lj} u(w_j) - a_l = \sum_{j=1}^{m} p_{ij} u(w_j) - a_i$$

Solution to the Principal's Problem

- Find λ such that

$$a_i = \lambda a_k + (1-\lambda) a_l$$

- By the CDCF

$$P_{ij} \leq \lambda P_{kj} + (1-\lambda) P_{lj}$$

- Note that

$$\sum_{j=1}^{m} p_{ij} u(w_j) - a_i = \sum_{j=1}^{m-1} P_{ij} \left(u(w_j) - u(w_{j+1}) \right) + u(w_m) - a_i$$

Solution to the Principal's Problem

- And so

$$\sum_{j=1}^{m} p_{ij} u(w_j) - a_i = \sum_{j=1}^{m-1} P_{ij} \left(u(w_j) - u(w_{j+1}) \right) + u(w_m) - a_i$$

$$\begin{aligned}
&\geq \lambda \left(\sum_{j=1}^{m-1} P_{kj} \left(u(w_j) - u(w_{j+1}) \right) + u(w_m) - a_k \right) \\
&\quad +(1-\lambda) \left(\sum_{j=1}^{m-1} P_{lj} \left(u(w_j) - u(w_{j+1}) \right) + u(w_m) - a_l \right) \\
&= \lambda \left(\sum_{j=1}^{m} p_{kj} u(w_j) - a_k \right) + (1-\lambda) \left(\sum_{j=1}^{m} p_{lj} u(w_j) - a_l \right)
\end{aligned}$$

Solution to the Principal's Problem

- This is impossible given

$$\sum_{j=1}^{m} p_{kj} u(w_j) - a_k > \sum_{j=1}^{m} p_{ij} u(w_j) - a_i$$

and

$$\sum_{j=1}^{m} p_{lj} u(w_j) - a_l = \sum_{j=1}^{m} p_{ij} u(w_j) - a_i$$

- Thus the solution to the constrained problem is the same as the solution to the original problem.