G5212: Game Theory

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Signalling

- We will now move to consider a simple model of costly signalling
- The classic example of this is the education signalling model by Spence [1973]
    - Different potential workers have different productivity levels
    - These productivity levels cannot be directly observed by firms
    - But workers can obtain education which can be observed
    - Under what circumstances can education be used by high productivity types to separate themselves from low productivity types?

The Basic Set Up

- Education is useless.
- But it is a pain to go through.
- And it is more painful if you have low ability.
- So studying signals your ability to employers.
  - A possible equilibrium is with employers offering a larger wage to people with a higher education.
  - Only highly productive people get a higher education.
  - But other equilibria exist, some of them very wasteful.
- Worker=principal; Employer=agent
  - Because worker ’proposes’ the contract

The Basic Set Up

- Two types of worker with different productivity given by θ
  - θ_2: High productivity
  - θ_1: Low productivity
- μ_1^*: Probability that the worker is of type θ_1
- Each worker can study for e years, after which they get a job that pays wage w
  - The wage they get may depend on their education
- Utility given by

$$u(w) - C(e, \theta)$$

The Basic Set Up

- We assume

$$u'(w) > 0 \quad u''(w) \leq 0$$

  - utility is increasing and weakly concave in w

- And

$$\frac{\partial C(e,\theta)}{\partial e} > 0 \quad \frac{\partial^2 C(e,\theta)}{\partial e^2} \leq 0$$
$$\frac{\partial C(e,\theta_1)}{\partial e} > \frac{\partial C(e,\theta_2)}{\partial e}$$

  - Costs are increasing in education
  - Marginal costs are higher for **lower** productivity individuals

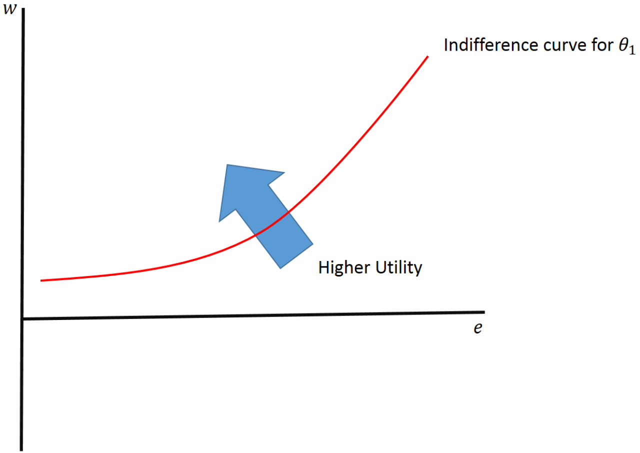
The Basic Set Up

- Indifference curves are given by

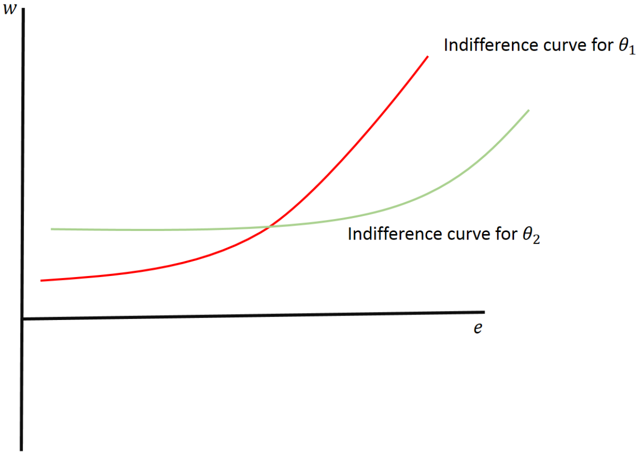
$$
\begin{aligned}
u'(w)dw - \frac{\partial C(e,\theta)}{\partial e}de &= 0 \\
\Rightarrow \quad \frac{dw}{de} &= \frac{u'(w)}{\frac{\partial C(e,\theta)}{\partial e}}
\end{aligned}
$$

- Upward sloping
- Higher utility to the North-west
- Always steeper for type θ_1
  - This is a **single crossing condition**

The Basic Set Up



The Basic Set Up



Firms

- Assumes that there are many firms
- Compete in the manner of Bertrand for the worker
- This means that wages will be equal to expected productivity given education level
- So if $\mu_1(e)$ is the probability that the worker is of type θ_1 having received education level e, the resulting wage level will be

$$w(e) = \mu_1(e)\theta_1 + (1 - \mu_1(e))\theta_2$$

First Best Solution

- What is the first best solution to this set up?
  - i.e. assuming that θ is observable?
- Each worker will receive a wage equal to their productivity regardless of the education level
- Will choose education level zero
- The **only** point of education in this set up is to signal quality

Second Best Solution

- This is a **dynamic game of incomplete information**
- So the appropriate solution concept is....
- Weak Perfect Bayesian Equilibrium

Definition

A strategy profile σ and a system of beliefs μ form a Weak Perfect Bayesian Equilibrium of an extensive game Γ_E if

1. σ is sequentially rational given μ
2. μ is derived from σ wherever possible

Second Best Solution

- We will focus on pure strategies
- So we need
  - An e_1^* and e_2^*: choice of education level for each type
  - A belief function $\mu_1(e) : \mathbb{R}_+ \rightarrow [0, 1]$ where $\mu_1(e)$ is the probability of type 1 given education level e
  - a wage function $w : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ were $w(e)$ is the wage paid at education level e
- Such that...

Second Best Solution

- Choice of education level is optimal given w

$$e_1^* \in \arg\max_{e \in \mathbb{R}_+} w(e) - C(e, \theta_1)$$

and

$$e_2^* \in \arg\max_{e \in \mathbb{R}_+} w(e) - C(e, \theta_2)$$

Second Best Solution

- Wage function is an equilibrium given beliefs

$$w(e) = \mu_1(e)\theta_1 + (1 - \mu_1(e))\theta_2$$

Second Best Solution

- Beliefs are formed using Bayes' rule where possible
  - if $e_1^* \neq e_2^*$

$$\begin{aligned} \mu_1(e_1^*) &= 1 \\ \mu_1(e_2^*) &= 0 \end{aligned}$$

  - if $e_1^* = e_2^*$

$$\mu_1(e_1^*) = \mu_1^*$$

Types of Solution

- It is going to turn out that there are **a lot** of solutions to this model
- The problem comes from the fact that the condition on beliefs only pins them down at one or two education levels
    - e_1^* and e_2^*
- Outside this, beliefs can be anything!
- We will describe two classes of solution
- Then we will use some equilibrium refinements to reduce the set of possible equilibria

Separating Equilibria

- The first type of equilibria are **separating equilibria**
- These are defined by the fact that the two types get different education levels
  - And so different wages
- What has to be true about a separating equilibrium?

Separating Equilibria

- Type θ_1 gets 0 education
  - Assume not
  - In equilibrium, it must be that

$$\mu_1(e_1^*) = 1$$

and so

$$w_1(e_1^*) = \theta_1$$

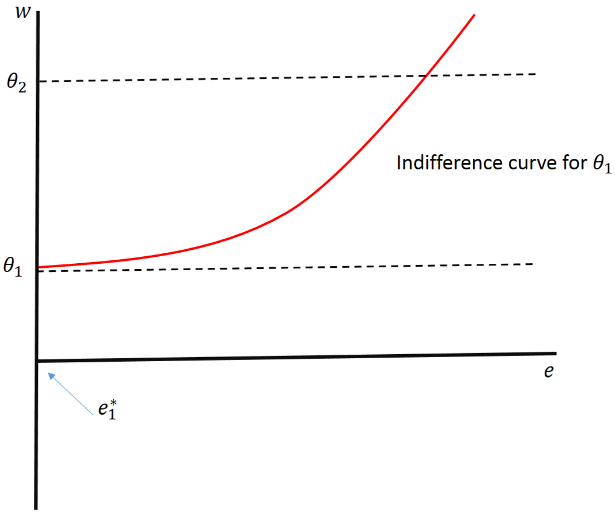
  - But notice that

$$\mu_1(0) \leq 1 \text{ and so } w_1(0) \geq \theta_1$$

  - Meaning

$$\begin{aligned} u(w(0)) - c(0, \theta_1) &\geq u(\theta_1) - c(0, \theta_1) > \\ u(\theta_1) - c(e_1^*, \theta_1) &= w_1(e_1^*) - c(e_1^*, \theta_1) \end{aligned}$$

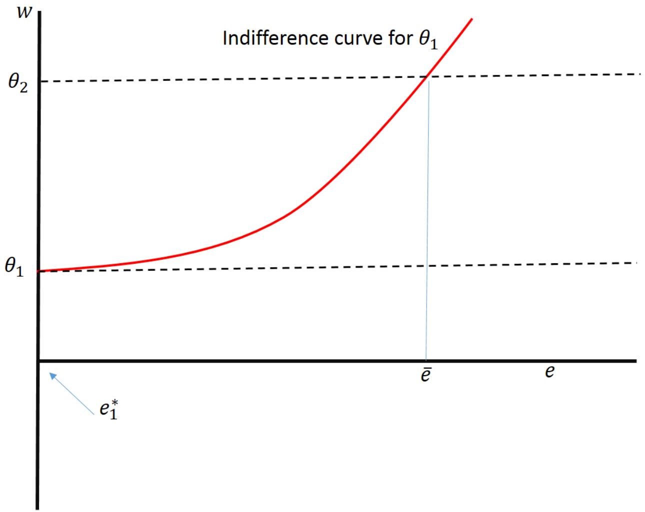
Separating Equilibria



Separating Equilibria

- Type θ_1 must not envy θ_2
  - In equilibrium
    $$\mu_1(e_2^*) = 0$$
    and so
    $$w_1(e_1^*) = \theta_2$$
  - It must be the case that θ_1 prefers 0 education and wage level θ_1, and so
    $$u(\theta_1) - c(0, \theta_1) \geq u(\theta_2) - c(e_2^*, \theta_1)$$
  - This puts a lower bound $\bar{e}$ on e_2^*

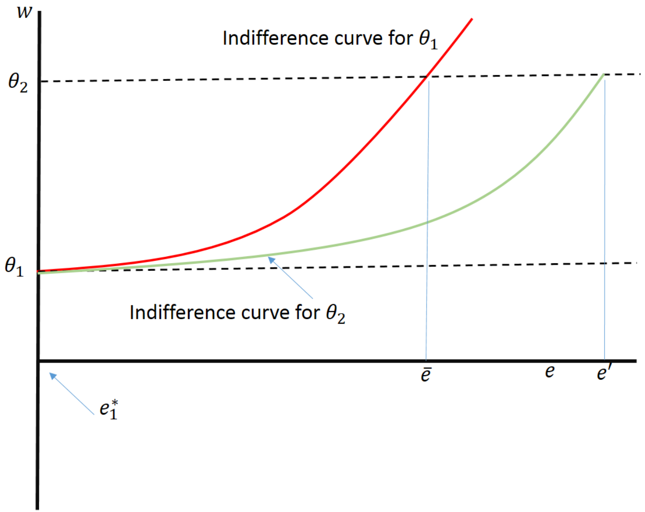
Separating Equilibria



Separating Equilibria

- Type θ_2 must not envy θ_1
  - In equilibrium
    $$\mu_1(e_2^*) = 0$$
    and so
    $$w_1(e_1^*) = \theta_2$$
  - It must be the case that θ_2 prefers e_2^* education and wage level θ_2, and so
    $$u(\theta_2) - c(e_2^*, \theta_2) \geq u(\theta_1) - c(0, \theta_2)$$
  - This puts an upper bound e' on e_2^*

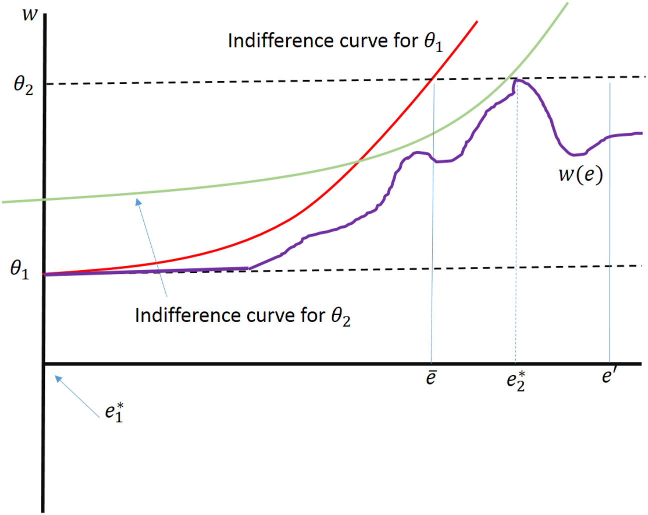
Separating Equilibria



Separating Equilibria

- Unfortunately this is all we can say
  - We can support any separating equilibrium with $e_2^* \in [\bar{e}, e']$ and $e_1^* = 0$
  - There are many wage functions which will support these equilibria
  - For example

Separating Equilibria



Pooling Equilibria

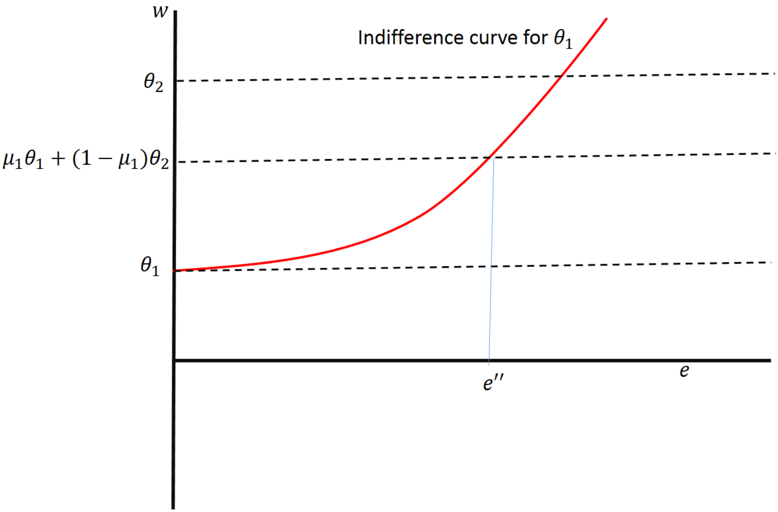
- The second type of equilibrium is a **pooling equilibrium**
- Here, both types of worker get the same level of education e^*
- Wages at this education level are

$$w(e^*) = \mu_1(e^*)\theta_1 + (1 - \mu_1(e^*))\theta_2$$

- Notice that e^* **need not** be 0
  - Though it is bounded above by e'' such that

$$\begin{aligned} & u((\mu_1\theta_1 + (1 - \mu_1)\theta_2)) - C(e'', \theta_1) \\ \geq \quad & u(\theta_1) - C(0, \theta_1) \end{aligned}$$

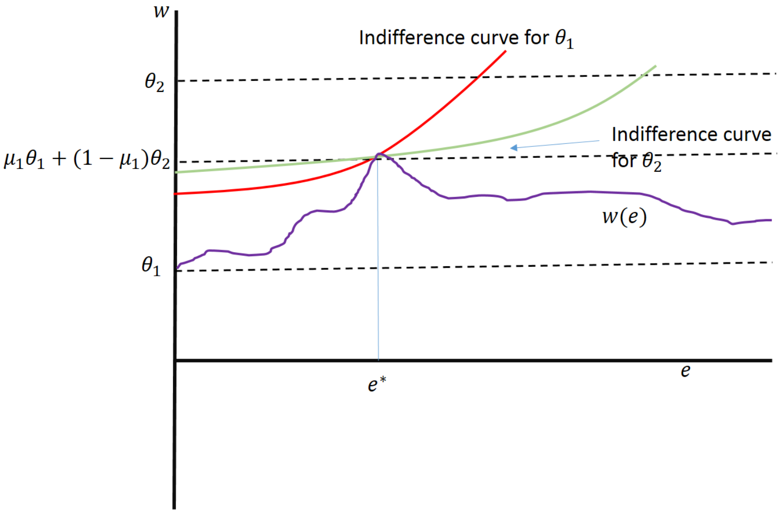
Pooling Equilibria



Pooling Equilibria

- if $e^* > 0$ then education is inefficient, and both types would be better off if it were banned
  - Same wage
  - Less expenditure on education
- Wage function must be such that both types prefer e^*

Pooling Equilibria



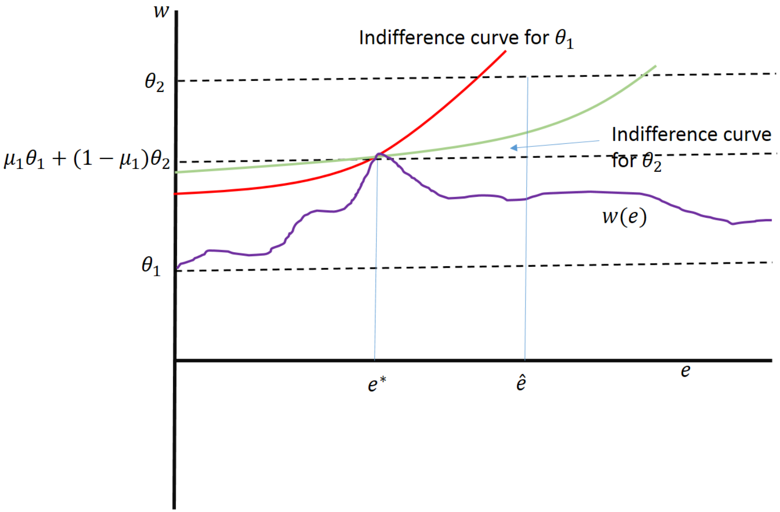
Multiple Equilibria

- The analysis so far is rather unsatisfying
- We have two types of equilibria
    - Pooling
    - Separating
- And within each type there are may equilibria indexed by the effort levels.
- The problem is the out of equilibrium beliefs
    - If no-one chooses an education level e then beliefs are not pinned down
    - Because beliefs are not pinned down they can be bad (i.e. high probability of θ_1)
    - If they are bad, no one chooses e
    - Self fulfilling prophecy

Multiple Equilibria

- Can we use some sensible further refinement on beliefs to rule out some of these equilibria?
- Luckily the answer is yes.
- Consider again the pooling equilibria we just looked at
- And think about the education level $\hat{e}$

The Intuitive Criterion



The Intuitive Criterion

- Notice that, as $w(\hat{e}) < \theta_2$, the employers must assign some probability that a low type would obtain that education level
- But is that sensible?
  - Imagine an employer who saw $\hat{e}$ being played.
  - They would be surprised, as no-one should play $\hat{e}$ in equilibrium
  - What should they think?

The Intuitive Criterion

- Notice, that, even if the pay at $\hat{e}$ was θ_2, the low type would prefer to play e^* and get wages $\mu_1\theta_1 + (1-\mu_1)\theta_2$, as

$$\begin{aligned} & w(\mu_1\theta_1 + (1-\mu_1)\theta_2) - C(e^*, \theta_1) \\ > \ & w(\theta_2) - C(\hat{e}, \theta_1) \end{aligned}$$

- Yet this is not true for the high type, as

$$\begin{aligned} & w(\mu_1\theta_1 + (1-\mu_1)\theta_2) - C(e^*, \theta_2) \\ < \ & w(\theta_2) - C(\hat{e}, \theta_2) \end{aligned}$$

- Thus it seems **really dumb** for the employers to put positive weight on the possibility that someone who plays $\hat{e}$ is of type θ_1
- But notice that if $\mu_1(\hat{e}) = 0$ and so $w(\hat{e}) = \theta_2$, the high type would rather play $\hat{e}$ than e^*

The Intuitive Criterion

- How do we formalize this insight?
- The intuitive criterion of Cho and Kreps [1987]
- First we need a definition of **equilibrium dominated**
- Broadly speaking, we say that a strategy s is equilibrium dominated for a type if they would prefer the equilibrium to that strategy even if they were treated in the best possible way following the play of s

Definition

A strategy s' for player i of type θ is dominated by an equilibrium strategy profile s if

$$u_i(s_i, s_{-i}, \theta) > \max_{\mu} u(s', s_{-i}(\mu), \theta)$$

where μ is the set of possible beliefs and $s_{-i}(\mu)$ is the best response function of the other players

The Intuitive Criterion

Definition

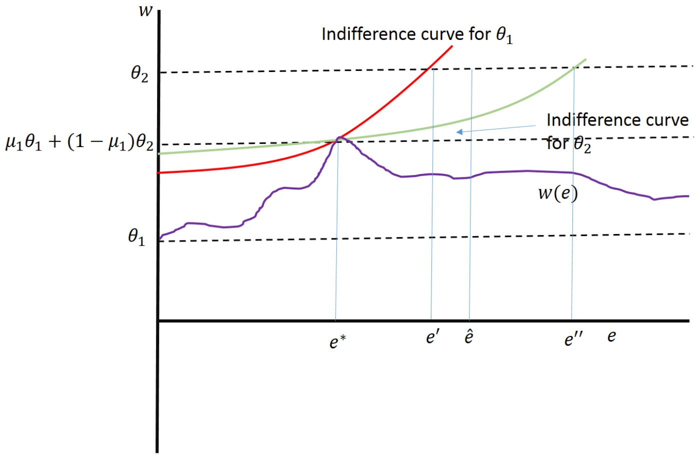
The intuitive criterion states that for any equilibrium and any message s not sent on the equilibrium path, if s is equilibrium dominated for some types but not others, beliefs must only place weight on those types that are not equilibrium dominated

- This has the flavor of a forward induction argument
- Imagine the following conversation:
  - Look, I am sending you this signal which is equilibrium-dominated for types A, B or C. But it is not so for types D and E. Therefore you cannot believe that I am types A, B or C.

The Intuitive Criterion

- Handily, the Intuitive Criterion kills all but one equilibrium for the signalling game we have been analyzing

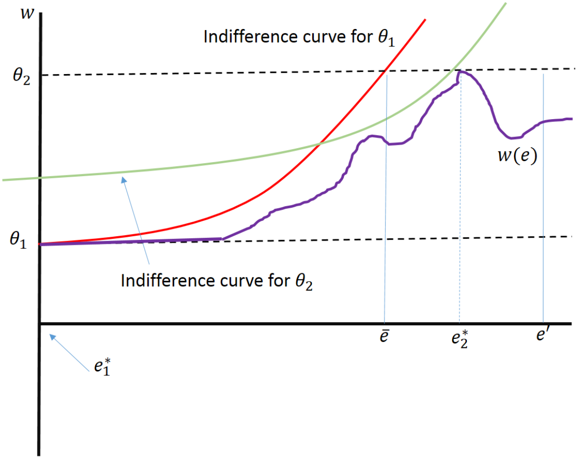
The Intuitive Criterion



The Intuitive Criterion

- For any pooling equilibrium we can always find a $\bar{e}$ which fails the intuitive criterion
- Any one between e' and e'' will do the trick:
    - e': Where the indifference curve for θ_1 through the equilibrium hits the θ_2 level
    - e'': Where the indifference curve for θ_2 through the equilibrium hits the θ_2 level
- What about separating equilibria?

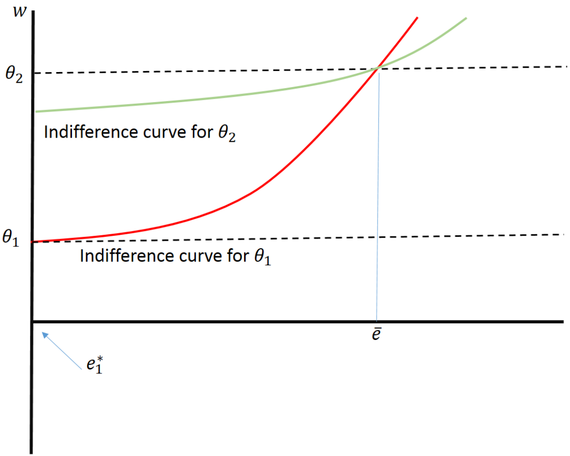
The Intuitive Criterion



The Intuitive Criterion

- In the above picture any e between $\bar{e}$ and e_2^* fails the intuitive criteria
- Only one separating equilibrium survives the refinement
- Which one?

The Intuitive Criterion



The Intuitive Criterion

- The minimum cost separating equilibrium survives the intuitive criterion
  - Any $e < \bar{e}$ is not equilibrium dominated for either type
  - Any $e > \bar{e}$ is equilibrium dominated for both types
- The intuitive criterion puts no further restrictions on beliefs