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Intermediate Microeconomics W3211

Lecture 2:
Indifference Curves and Utility

Columbia University, Spring 2016

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Introduction

What do your preferences look like?

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The Story So Far....

- In Monday's lecture we described how we would model consumer choice as **constrained optimization:**

1. **CHOOSE a consumption bundle**
2. **IN ORDER TO MAXIMIZE preferences**
3. **SUBJECT TO the budget constraint**

- Consumption bundles and budget constrains were dealt with fairly thoroughly
- Preferences may still seem a bit mysterious
- Today, we will attempt to de-mystify

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Today's Aims

1. Describe how to represent preferences on our handy graphs
   - **Indifference Curves**
2. Describe how to represent preferences using maths
   - **Utility functions**
3. Describe 'standard' preferences
   - Monotonic
   - Convex
4. Introduce some other handy classes of preferences
   - Perfect substitutes
   - Perfect compliments
   - Cobb Douglas

- Reminder: Varian Ch. 3 & 4, Feldman and Serrano Ch 2

Indifference Curves

Or: how to see your preferences

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Going from Three Dimensions to Two

- Think back to our nice, simple example with two goods
- Made it nice and easy to graph the consumption bundles and budget constraints

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Going from Three Dimensions to Two

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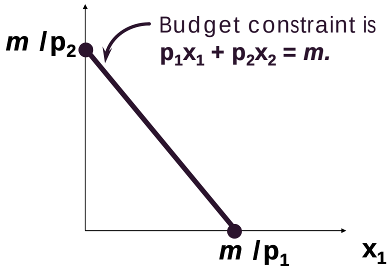
- Made it nice and easy to graph the consumption bundles and budget constraints



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Going from Three Dimensions to Two

- Unfortunately, we now have **three** pieces of information associated with each bundle
  - Amount of good 1
  - Amount of good 2
  - Whether this bundle is preferred or not to others
- How can we represent this information?
- One way would be to use three dimensions:
  - Bundles that are more preferred are placed higher on dimention 3 than those which are less preferred

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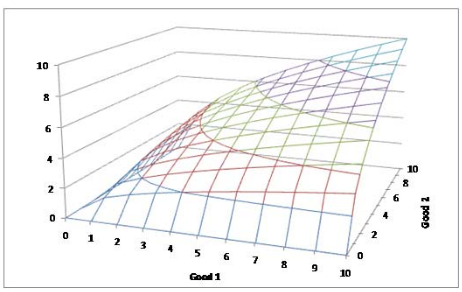
Going from Three Dimensions to Two



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Going from Three Dimensions to Two

- While this would work, graphing in three dimensions is a pain
- As we will see, we can get a lot more intuition if we can work in two dimensions
- So how can we go from three dimensions to two?
- Luckily, cartographers have the answer

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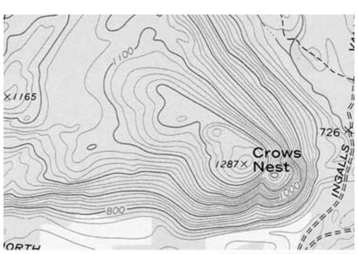
Going from Three Dimensions to Two



US Geological Survey map

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Going from Three Dimensions to Two

- On a map, **contour lines** link areas of equal **height**
- We will use **indifference curves** which link areas of equal **preference**

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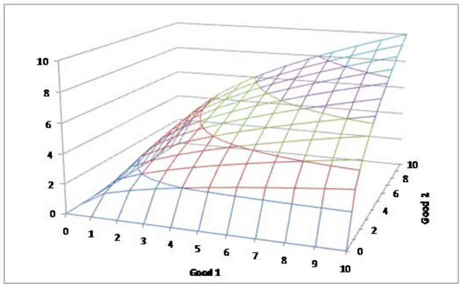
Going from Three Dimensions to Two



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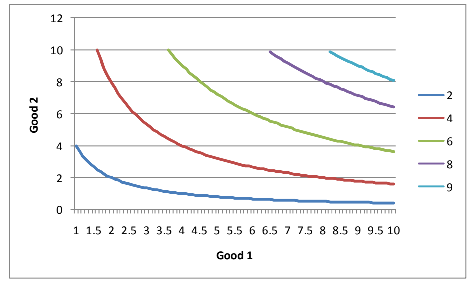
Going from Three Dimensions to Two



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Indifference Curves

- A simple example
- 2 goods:
  - Apples
  - Oranges
- Suppose our agent likes **both**
- Take some bundle x'
- An indifference curve links all bundles of goods which are indifferent to x'
- For example, say that $x' \sim x'' \sim x'''$

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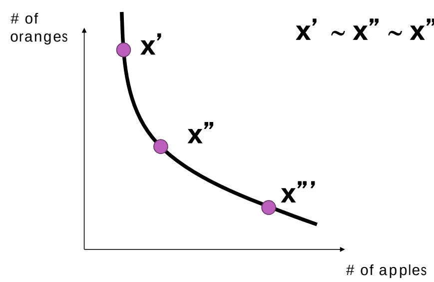
Graphically

$x' \sim x'' \sim x'''$



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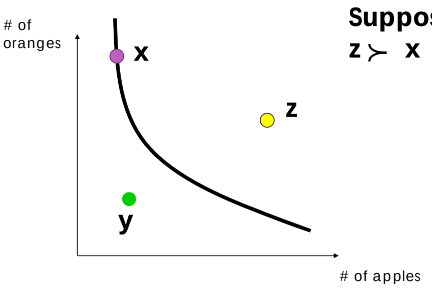
Indifference Curves

Suppose

$z \succ x \succ y$



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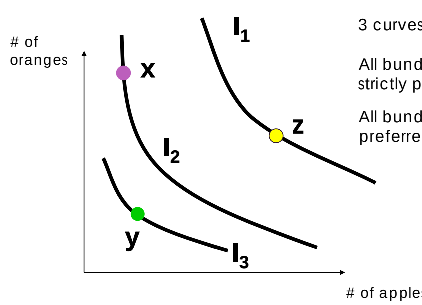
Indifference Curves



3 curves

All bundles in I_1 are strictly preferred to all in I_2

All bundles in I_2 are strictly preferred to all in I_3.

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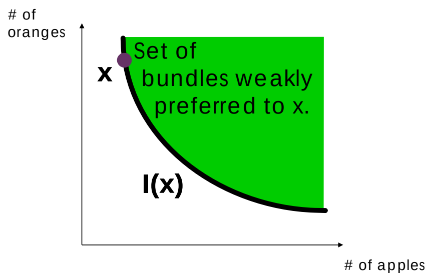
Indifference Curves



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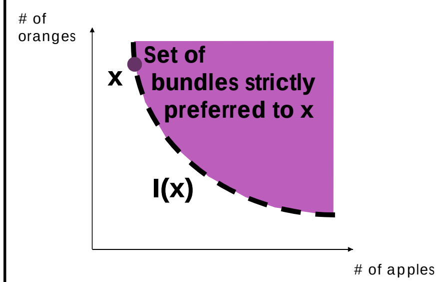
Indifference Curves



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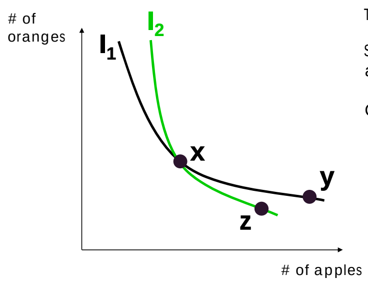
Can indifference curves intersect?



Take 2 curves I_1 and I_2

Suppose $z \prec y$ so these are different curves

Can they intersect?

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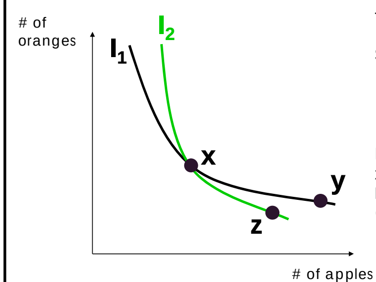
Can indifference curves intersect?



Take 2 curves I_1 and I_2

Suppose $z \prec y$ so these are different curves

Can they intersect?

No! From $\mathbf{I_1}$, $x \sim y$. From I_2, $x \sim z$.
From transitivity $y \sim z$.
Contradiction!!

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The Marginal Rate of Substitution

- One crucial thing we will want to know about preferences:

"If I take away one apple, how many oranges would I have to give you to keep you indifferent"

- Or more accurately, at what rate do I have to change oranges for apples to keep you indifferent?
- This is the **marginal rate-of-substitution** (MRS) between apples and oranges
- Mathematically

$$MRS(x_a, x_o) = -\lim_{\Delta x_a \to 0} \frac{\Delta x_o}{\Delta x_a}$$

Such that $(x_a, x_o) \sim (x_a + \Delta x_a, x_o + \Delta x_o)$

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The Marginal Rate of Substitution

- Why is this of interest?
- At this stage you can take my word for it…
- ….or you can think about the following
- "If the rate at which I am willing to trade off apples to oranges is **higher** than the relative price of apples and oranges, am I maximizing my preferences"?
- WARNING: Sometimes MRS is defined without the minus sign
  - There is no consensus about which is correct
  - I will accept either definition from you, as long as you are consistent!

The Marginal Rate of Substitution

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- Claim: It is very easy to see the marginal rate of substitution from the indifference curve.
- The MRS at a particular point is the negative of slope of the indifference curve at that point

Marginal Rate of Substitution

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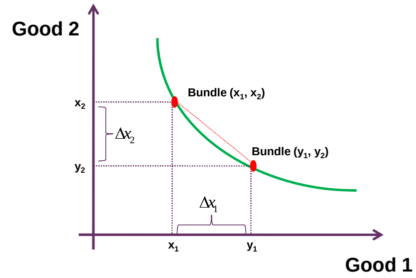

Marginal Rate of Substitution

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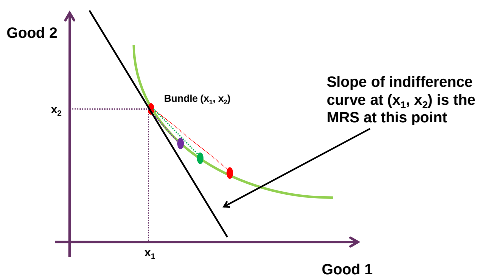

Preferences and Utility

Or: Your preferences in numbers

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Where are the Numbers?

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- So far we have represented the **objectives** in our consumer's problem with preferences and weird symbols.
- I bet you are crying out to get away from these symbols and use some proper, god fearing numbers to represent what the consumer wants to maximize
- This would allow us to do lots of cool things
  - For example take derivatives
- Moreover, you have probably heard of the concept of a 'utility function'
  - Reports how 'happy' a particular bundle of goods makes someone
- So why can't we work with utility functions?

Where are the Numbers?

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- Okay, so we **are** going to work with utility functions
- But first I want you to understand how utility functions are used in economics
- This is slightly counterintuitive…
- …but useful for you to understand what is going on
- For many people this is the most confusing bit of the course, so hold on!

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In the beginning….

- Back in the days of Marshall (1890's) utility was considered to be a real, measurable, cardinal scale
- Lurking in people's brains was the 'happiness', or utility, associated with different bundles of goods
- People made choices in order to maximize this happiness
- The nature of utility could be derived from first principles
  - Calories
  - Water
  - Warmth
  - Etc.

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In the beginning….

- Three problems with this approach

1. What evidence did we have that there *was* utility lurking in the brain
   - No way to directly measure it
2. All attempts to derive utility from first principles (and so explain choice) failed
3. What did it mean for the utility of bundle x to be twice that of bundle y?

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20th Century: Preferences!

- This lead to a change of approach in 20th century
- Preferences were taken as the primitive thing that people maximized
- Why? What is the big advantage of preferences over utility?
- We can plausibly **measure** preferences in a way we cannot measure utility
- How?
- One good way would be through choice
  - x is preferred to y if x is chosen over y
- **Side note:** Neuoscientists and neuro-economists are trying to take us back to the days of Marshall

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Utility from Preferences

- Despite this realization, it is still useful to be able to work with utility functions
  - For one thing, we can take derivatives!
- So the question became whether we can build a utility function from preferences
- Let's start with a set of preferences $\succsim$ on different bundles of goods
- Can we build a utility function u that **represents** (or contains the same information as) $\succsim$
- i.e. we can find a way of assigning utility numbers to bundles such that, for any two bundles x and y

$x \succsim y$ if and only if $u(x) \geq u(y)$

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Utility from Preferences

- If we can do this, then we can 'pretend' that the consumer is maximizing utility
- Maximizing preferences: choosing x such that $x \succsim y$ for all available y
- Maximizing utility: choosing x such that $u(x) \geq u(y)$ for available y
- These are the same thing **if** the utility function represents the preferences: i.e.

$x \succsim y$ if and only if $u(x) \geq u(y)$

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When do we have a utility function?

- So, when does a preference relation allow a utility representation?
- Answer: as long as it is well behaved!
  - Complete
  - Transitive
  - Reflexive
  - (also, if the set of available options is not countable, we need continuity, but don't worry so much about this)

When do we have a utility function?

- If preferences are not well behaved, will there be a utility representation?
- No!
- For example: failure of transitivity
  - $x \succsim y$, $y \succsim z$ but NOT $x \succsim z$
  - Say u represents these preferences
  - $x \succsim y$ implies $u(x) \geq u(y)$
  - $y \succsim z$ implies $u(y) \geq u(z)$
  - By the power of maths, this implies $u(x) \geq u(z)$
  - But if u represents the preferences this would imply $x \succsim z$
  - Contradiction
- A preference relation allows a utility representation if and only if it is well behaved

What do Utility Numbers Mean?

- Lets say we have
  - Three alternatives x,y and z
  - Preferences $\succsim$ over x,y and z
  - A utility function u that represents these preferences

$$u(x) = 1$$
$$u(y) = 2$$
$$u(z) = 3$$

- Does it mean anything that the utility of z is three times the utility of y?
- Not really: Consider instead the utility numbers

$$u(x) = 1$$
$$u(y) = 1.5$$
$$u(z) = 2$$

- These represent the same preferences
  - Would lead to the same choices
- But now the utility of z is twice that of y

What do Utility Numbers Mean?

- Utility numbers only have **ordinal** meaning
  - The ordering of the numbers matters
- They do not have **cardinal** meaning
  - The difference between the numbers does not matter
- This is another way of saying that **many utility functions** represent the same preferences
- An obvious question: can we determine the relationship between utility functions that represent the same preferences?

What do Utility Numbers Mean?

- **Theorem:** Take two utility functions u and v. They both represent the same preferences if and only if there is a strictly increasing function f such that

$$\boldsymbol{v(x) = f(u(x))}$$

for all x

- For example say u represents preferences and $v(x) = 2u(x) + 3$

$x \succsim y$ if and only if

$u(x) \geq u(y)$ if and only if

$2u(x) + 3 \geq 2u(y) + 3$ if and only if
$v(x) \geq v(y)$

Utility Functions & Indifference Curves

- We now have two ways of representing preferences
  - Indifference curves
  - Utility functions
- What is the relationship between them?
- An indifference curve links equally preferred bundles.
- Equal preference $\Longrightarrow$ same utility level.
- Therefore, all bundles in an indifference curve have the **same utility level**
- **To each indifference curve we have one utility level**

Utility Functions & Indifference Curves

- Consider bundles (4,1), (2,2) and (2,3)
- Suppose (4,1) indifferent to (2,2) and that (2,3) is preferred to both
- Suppose that (4,1) and (2,2) are in the indiff. curve with utility level U ≡ 4
- And bundle (2,3) is in the indiff. curve with utility level U =6.

Utility Functions & Indifference Curves

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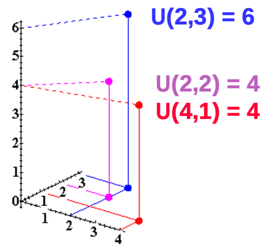
Plotting the utility on the vertical axis



Graphically

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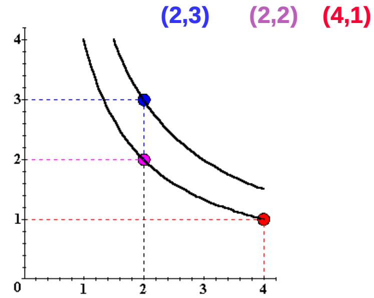

From utility to indifference curves

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- Suppose we want to draw the indifference curves for a particular utility function
  $$u(x_1, x_2) = x_1 x_2$$
- Indifference curves are elements with the same utility
- So? Just set the utility equal to k

$$x_1 x_2 = k \text{ implies}$$
$$x_2 = \frac{k}{x_1}$$

- This is the equation for the indifference curve for various k

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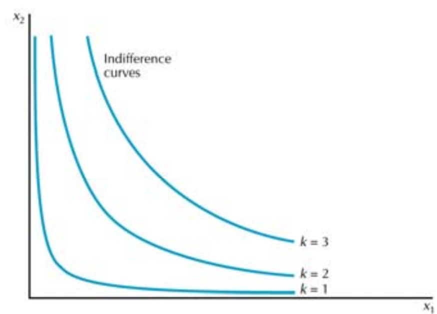

An Intelligence test

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- Another example
- Consider $U(x_1, x_2) = x_1^2 x_2^2$
- What do indifference curves look like?
- We can proceed as before

An Intelligence test

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- Another example
- Consider $U(x_1, x_2) = x_1^2 x_2^2$
- What do indifference curves look like?
- We can proceed as before
- Or: we can **smart it out**
- Look: $U(x_1, x_2) = x_1^2 x_2^2 = (x_1 x_2)^2$
- But: $f(y) = y^2$ is a strictly increasing function
- Thus: These are the same preferences as $U(x_1, x_2) = x_1 x_2$

Marginal Utilities and The Marginal Rate of Substitution

- Recall, that I claimed that we would be interested in the **Marginal Rate of Substitution**
  - Slope of the indifference curve
  - Rate at which you would change one good for another while remaining indifferent
- It will be very handy to know how to calculate the MRS from utility functions
- To do so we need to introduce the idea of **Marginal Utility**

Marginal Utility

- Marginal Utility: the rate at which utility changes with the quantity of one good, keeping the other constant
- The partial derivative of the utility function

$$MU_i = \frac{\partial u(x)}{\partial x_i}$$

- So, for example, if $u(x_1, x_2) = x_1^{\frac{1}{2}} x_2^2$ then

$$MU_1 = \frac{\partial u(x_1,x_2)}{\partial x_1} = \frac{1}{2} x_1^{-\frac{1}{2}} x_2^2$$

$$MU_2 = \frac{\partial u(x_1,x_2)}{\partial x_2} = 2 x_1^{\frac{1}{2}} x_2$$

Marginal Utility and MRS

- So what is the relationship between marginal utility and MRS?
- Recall that the MRS is the negative of the slope of the indifference curve
- And that the indifference curve is defined by $u(x_1, x_2) = k$ for some k.
- In order to get the slope, take the total derivative

$$\frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2 = 0$$

Which implies

$$\frac{dx_2}{dx_1} = -\frac{\partial u}{\partial x_1} \Big/ \frac{\partial u}{\partial x_2}$$

Marginal Utility and MRS

- And so

$$MRS(x_1, x_2) = -\frac{dx_2}{dx_1} = \frac{\partial u}{\partial x_1} \Big/ \frac{\partial u}{\partial x_2}$$

- Marginal Rate of Substitution equals the ratio of marginal utilities
- Say for example $u(x_1, x_2) = x_1 \ x_2$,and so

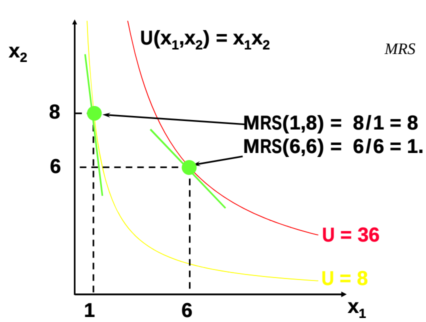
$$MU_1 = \frac{\partial u(x_1,x_2)}{\partial x_1} = x_2$$

$$MU_2 = \frac{\partial u(x_1,x_2)}{\partial x_2} = x_1$$

- Implying

$$MRS(x_1, x_2) = \frac{x_2}{x_1}$$

Graphically



$$MRS = \frac{x_2}{x_1}$$

Are Marginal Utilities and MRS Meaningful?

- Recall that utility function is not unique
- I can "transform" it and still represent the same preferences
- But notice: marginal utility depends on the utility used
- Thus: if I transform the utility, I will change the marginal utility as well
- This means: **marginal utility has little behavioral content**
- **Don't read too much into it!**
- E.g., if I double the utility I double also the marginal utility but **behaviorally identical**!
- This is a subtle point: make sure you understand it

Monotonic Transformations & Marginal Rates-of-Substitution

- What about the MRS?
- For $U(x_1,x_2) = x_1x_2$ the MRS = x_2/x_1.
- Create $V = U^2$; *i.e.* $V(x_1,x_2) = x_1^2x_2^2$. What is the MRS for V?

$$MRS = \frac{dv}{dx_1}\Big/\frac{dv}{dx_2} = 2x_1x_2^2\big/2x_1^2x_2 = x_2/x_1$$

which is the same as the MRS for U

Monotonic Transformations & Marginal Rates-of-Substitution

- More generally, if V = f(U) where f is a strictly increasing function, then

$$MRS = \frac{dv}{dx_1}\Big/\frac{dv}{dx_2} = \frac{f'(u(x))\frac{du}{dx_1}}{f'(u(x))\frac{du}{dx_2}} = \frac{\frac{du}{dx_1}}{\frac{du}{dx_2}}$$

MRS unchanged by a positive monotonic transformation

Monotonic Transformations & Marginal Rates-of-Substitution

- This is an extremely important point
- Marginal Utility – depends on the specific utility function used
  - Not very meaningful
- Marginal Rate of Substitution – does not depend on the specific utility function used
  - All utility functions which represent the same preferences give the same answer
  - Behaviorally meaningful

Summary

Summary

- Today we have described two ways of representing preferences
  - Indifference curves (and the associated concept of MRS)
  - Utility functions
- Described the (somewhat confusing) relationship between preferences and utility
  - Utility used to represent preferences
  - Are not unique – many utility functions represent the same preferences (and lead to the same choices)
- Shown how to calculate MRS from utility functions
  - Marginal utility depends on the precise form on utility used
  - MRS does not