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Intermediate Microeconomics W3211

Lecture 5: Choice and Demand

Columbia University, Spring 2016

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Introduction

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The Story So Far….

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- We have now have had a first attempt at solving the consumer problem
- Provides a recipe based on finding three types of solution
  - Corner solutions
  - Tangency points
  - Kinks

Today's Aims

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1. Provide some more mathematically sophisticated tools to find tangency points
   - Varian Ch. 5 appendix, Feldman and Serrano Ch. 3 appendix
2. Introduce the concept of a demand function, which measures how the amount of good a consumer purchaces changes with
   - Prices
   - Income
   - Varian Ch. 6, Feldman and Serrano Ch. 4
3. Discuss how demand changes with income
   - Varian Ch. 6, Feldman and Serrano Ch. 4

Calculus and the Consumer's Problem

1: Constrained to unconstrained problems

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Solving the Consumer's Problem

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- One of the reasons that calculus is so useful is that it allows us to find the optimal solutions for constrained and unconstrained optimization problems
- This relies on derivatives
- Only works for finding points of tangency
  - Not corner solutions
  - Not kinks
- So these tools are useful, but don't forget you still need to worry about other solutions

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Calculus: A Reminder

- Imagine you have an **unconstrained** optimization problem

  Choose x to maximize f(x)

- How can you use calculus to solve this?

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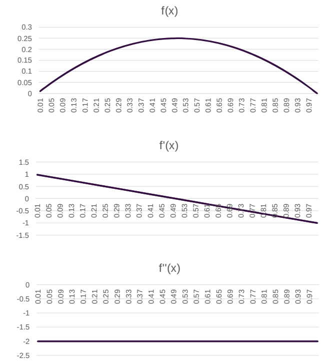
Calculus: A Reminder



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Calculus: A Reminder

- A maximum of the function will occur where it is ‘flat’
- i.e. where the derivative is zero
- Is it always the case that the derivative being zero means that you have found a maximum?

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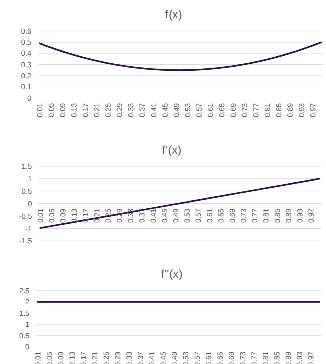
Calculus: A Reminder



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Calculus: A Reminder

- Flat points of the function occur either at **maxima** or **minima**
- To differentiate between the two, check the second derivative
- If you have found a point at which
  - The first derivative is zero
  - The second derivative is negative
- Then you have found a **local** maximum
- However you still have to worry about
  - Other local maxima

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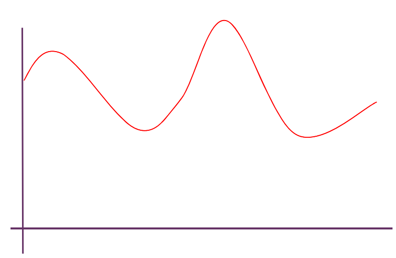
Calculus: A Reminder



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  - Other local maxima
  - Corner solutions

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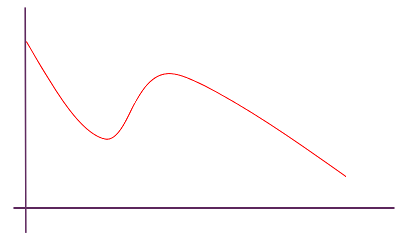
Calculus: A Reminder



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  - Other local maxima
  - Corner solutions
  - Kinks

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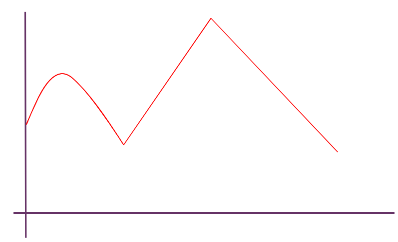
Calculus: A Reminder



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From Constrained to Unconstrained Problems

- How does this help us?
- Tells us how to solve **unconstrained** optimization problems
- But we have a **constrained** optimization problem

Choose x_1, x_2 to Maximize $u(x_1, x_2)$

Subject to $p_1x_1 + p_2x_2 = m$

- Answer: We can substitute in using the budget constraint to make it an unconstrained problem
- Note – in order to do so we are assuming the budget constrain holds with equality
  - Monotonic preferences

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From Constrained to Unconstrained Problems

- An example:

Choose x_1, x_2 to Maximize $u(x_1, x_2)$=x_1x_2

Subject to $p_1x_1 + p_2x_2 = m$

- Using the budget constraint we get

$$x_2 = \frac{m}{p_2} - \frac{p_1}{p_2}x_1$$

- Substitute this into the utility function

$$x_1x_2 = x_1\left(\frac{m}{p_2} - \frac{p_1}{p_2}x_1\right)$$

- Can choose any x_1 (such that x_1 and x_2 are greater than or equal to zero) to maximize utility
- Don't have to worry about the budget constraint
- x_2 adjusts automatically to changes in x_1

From Constrained to Unconstrained Problems

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- Problem becomes

$$\text{Choose } x_1 \text{ to Maximize } x_1x_2 = x_1\left(\frac{m}{p_2} - \frac{p_1}{p_2}x_1\right)$$

- Taking derivatives gives

$$f'(x_1) = \frac{m}{p_2} - 2\frac{p_1}{p_2}x_1$$

$$f''(x_1) = -2\,\frac{p_1}{p_2}$$

- Second derivative is negative, so first order condition will give us a local maximum

$$\frac{m}{p_2} - 2\frac{p_1}{p_2}x_1 = 0$$

- Implies (using the budget constraint for x_2)

$$\frac{m}{2p_1} = x_1 \text{ and } \frac{m}{2p_2} = x_2$$

The Advantages of This Approach

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- Unsurprisingly, gives the same answer as the recipe from last week
- What is the advantage of this approach?
- One thing is the second order conditions
- Helps us to identify when we have found a local minima
- Consider **concave** preferences

$$u(x_1, x_2) = x_1^2 + x_2^2$$

- You should check you agree, but these preferences give utility that looks like…

The Advantages of This Approach

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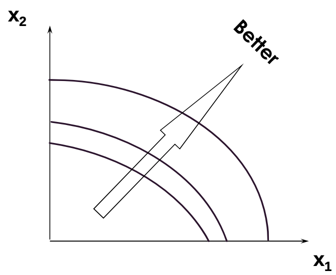

The Advantages of This Approach

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- So a tangency point is actually a minimum
- The second order conditions can help us spot this

$$\text{Choose } x_1, x_2 \text{ to Maximize } u(x_1, x_2) = x_1^2 + x_2^2$$

$$\text{Subject to } p_1x_1 + p_2x_2 = m$$

- Becomes

$$\text{Choose } x_1 \text{ to Maximize } x_1x_2 = x_1^2 + \left(\frac{m}{p_2} - \frac{p_1}{p_2}x_1\right)^2$$

- Taking derivatives gives

$$f'(x_1) = 2x_1 - 2\frac{p_1}{p_2}\left(\frac{m}{p_2} - \frac{p_1}{p_2}x_1\right)$$

$$f''(x_1) = 2 + 2\left(\frac{p_1}{p_2}\right)^2 > 0$$

- Second order condition fails, tells us we have a maximum

Calculus and the Consumer’s Problem

2: Karush Kuhn Tucker – The ultimate tool for constrained optimization problems

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Karush Kuhn Tucker

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- Mathematicians have developed even more powerful tools for solving constrained optimization problems
- We won’t have time to go over them thoroughly in this class
- But I want you to be aware of them, for two reasons
  1. They can make it easier to solve problems we will deal with
  2. You will certainly need them if you take economics further
- You will not need to use these techniques to solve questions in exams
  - But you should use them if you want
  - You may need them to solve homework questions

Karush Kuhn Tucker

- Imagine you have a constrained optimization problem

Choose x_1, x_2 to Maximize $u(x_1, x_2)$

Subject to $\mathrm{f}(x_1, x_2) = 0$

- (in our case $\mathrm{f}(x_1, x_2) = p_1 x_1 + p_2 x_2 - m$)
- It turns out that you can solve this problem by solving a *related* unconstrained maximization problem

Choose x_1, x_2, μ to Maximize
$$L(x_1, x_2, \mu) = u(x_1, x_2) - \mu \mathrm{f}(x_1, x_2)$$

- μ is the **Lagrange Multiplier**
- You can treat it just like another thing to choose (like x_1 and x_2)

Karush Kuhn Tucker

Choose x_1, x_2, μ to Maximize
$$L(x_1, x_2, \mu) = u(x_1, x_2) - \mu \mathrm{f}(x_1, x_2)$$

- How do we solve this problem?
- Treat it like any unconstrained optimization problem
- Take first derivatives and set them to zero!

Karush Kuhn Tucker

- A worked example:

Choose x_1, x_2 to Maximize $u(x_1, x_2) = x_1 x_2$

Subject to $p_1 x_1 + p_2 x_2 = m$

- First set up the Lagrange Function
$$L(x_1, x_2, \mu) = x_1 x_2 - \mu(p_1 x_1 + p_2 x_2 - m)$$
- Then take derivatives

Karush Kuhn Tucker

$$L(x_1, x_2, \mu) = x_1 x_2 - \mu(p_1 x_1 + p_2 x_2 - m)$$

- Then take derivatives
$$\frac{\partial L}{\partial x_1} = x_2 - \mu p_1 = 0$$
$$\frac{\partial L}{\partial x_2} = x_1 - \mu p_2 = 0$$
$$\frac{\partial L}{\partial \mu} = p_1 x_1 + p_2 x_2 - m = 0$$
- First two equations gives the (very familiar looking!)
$$\frac{x_2}{x_1} = \frac{p_1}{p_2}$$
- Last equation gives the (very familiar looking)
$$p_1 x_1 + p_2 x_2 = m$$

Advantages of Karush Kuhn Tucker

- Why have I bothered to tell you this? What is the advantage of this approach?
- Imagine that we had three goods

Choose x_1, x_2, x_3 to Maximize $u(x_1, x_2, x_3)$

Subject to $p_1 x_1 + p_2 x_2 + p_3 x_3 = m$

- Substitution method won't work (we have too many unknowns)
- Can't draw indifference curves in two dimensions
- However the KKT approach will still work

Karush Kuhn Tucker

$$L(x_1, x_2, x_3, \mu) = u(x_1, x_2, x_3) - \mu(p_1 x_1 + p_2 x_2 + p_3 x_3 - m)$$

- Then take derivatives
$$\frac{\partial L}{\partial x_1} = \frac{\partial u}{\partial x_1} - \mu p_1 = 0$$
$$\frac{\partial L}{\partial x_2} = \frac{\partial u}{\partial x_2} - \mu p_2 = 0$$
$$\frac{\partial L}{\partial x_3} = \frac{\partial u}{\partial x_3} - \mu p_3 = 0$$
$$\frac{\partial L}{\partial \mu} = p_1 x_1 + p_2 x_2 + p_3 x_3 - m = 0$$
- Four equations, four unknowns – will generally be able to find a solution

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A Word of Warning

- Are KKT solutions always the solutions to the consumer's problem?
- No!
- KKT conditions find points of tangency
- But the usual caveats apply
  - Tangency are neither necessary or sufficiency for optimality
- Solutions may still be at corners or kinks, or preferences may be non-monotonic
- Also, KKT conditions may pick out minima instead of maxima
  - Just as when we use differentiation to solve unconstrained optimization problems
  - There are equivalent second order conditions, but you don't need to worry about them for this course
- Remember this!
  - I **will** try to fool you, and I **will** catch some of you!

Demand

Or, the effect of prices and income

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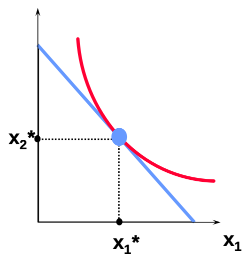
Defining Demand

- We now know how to solve the consumer's problem



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Defining Demand

- This solution depends on budget constraint
  - The **prices** of the two goods, and the **income**
- We define the amount that the consumer will buy, given prices and incomes, as their **demand**

  $$x_1(p_1, p_2, y)$$
- Is the amount that the consumer will buy of good one if prices are p_1 and p_2 and income is y
- This is the demand function for good 1
- Similarly, $x_2(p_1, p_2, y)$ is the demand function for good 2
- More generally, $x_i(p, y)$ is the demand function for good i if the vector of prices p
- In the next half a lecture, we will think about how demand changes with prices and income

Demand

1: The effect of income on demand

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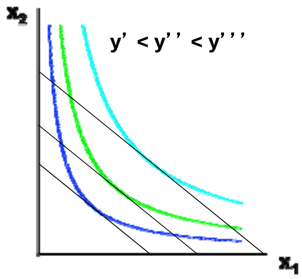
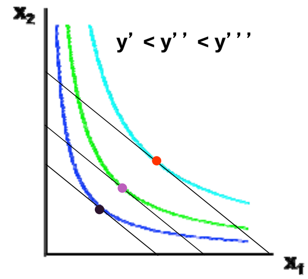
How Does Demand Change with Income?

- Question 1: how does $x_1(p_1, p_2, y)$ change with y?
  - Keeping prices fixed

Income Changes



Income Changes



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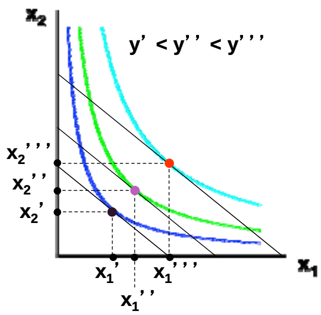

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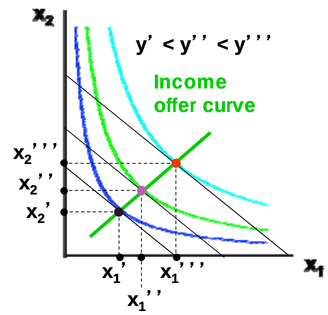
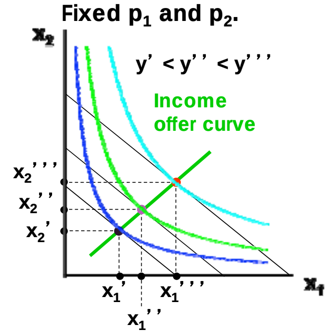


Income Changes

- The curve that unites the optimal choices for various levels of income is called **income offer curve or income expansion path**
  - Axes are x_1 and x_2
- A plot of quantity demanded against income is called an **Engel curve.**
  - Axes are x_1 and y
  - Traditionally y is on the vertical axis

Income Changes

Fixed p_1 and p_2.



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Income Changes

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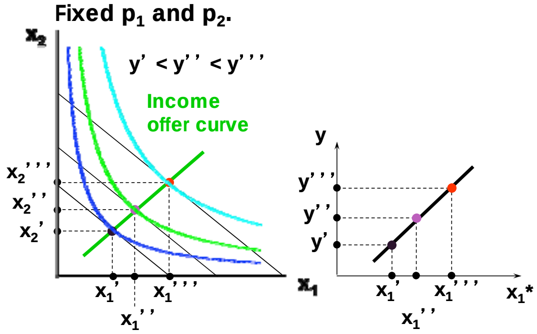

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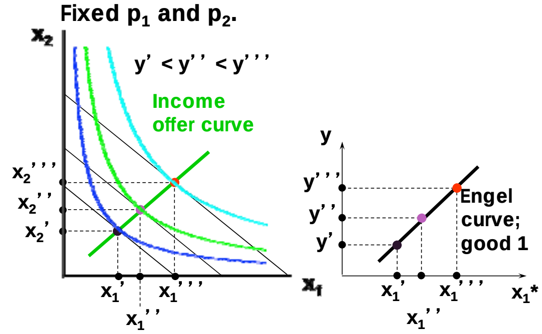

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Income Changes

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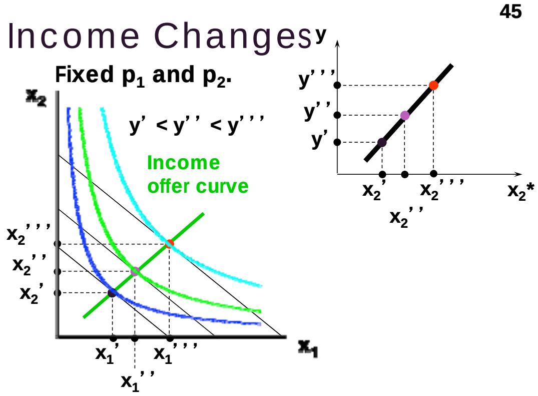

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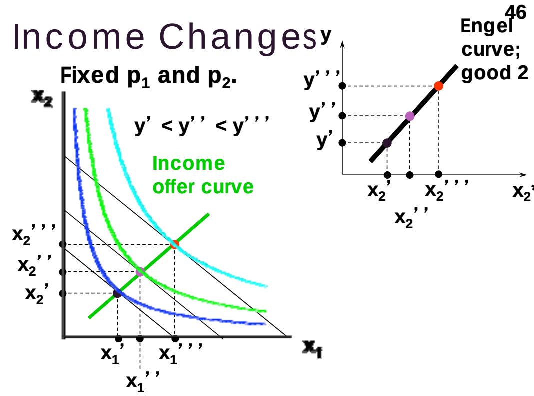

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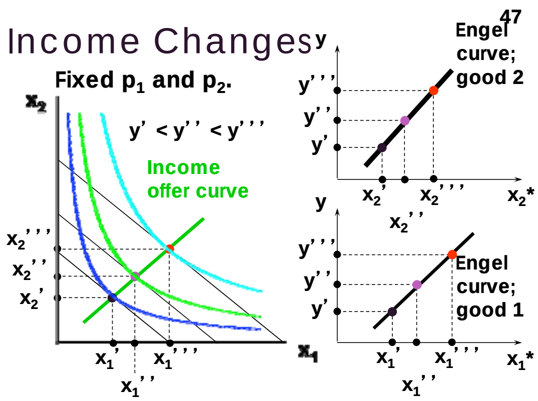

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Income Changes and Cobb-Douglas Preferences

- An example of computing the equations of Engel curves; the Cobb-Douglas case.

$$U(x_1, x_2) = x_1^a x_2^b.$$

- The ordinary demand equations are

$$x_1^* = \frac{ay}{(a+b)p_1}; \quad x_2^* = \frac{by}{(a+b)p_2}.$$

Income Changes and Cobb-Douglas Preferences

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$$x_1^* = \frac{ay}{(a+b)p_1}; \quad x_2^* = \frac{by}{(a+b)p_2}.$$

Rearranged to isolate y, these are:

$$y = \frac{(a+b)p_1}{a} x_1^*$$ **Engel curve for good 1**

$$y = \frac{(a+b)p_2}{b} x_2^*$$ **Engel curve for good 2**

Income Changes and Cobb-Douglas Preferences

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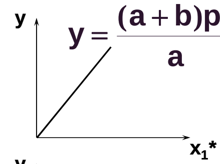
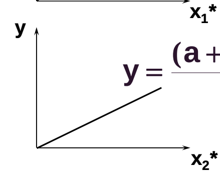
$$y = \frac{(a+b)p_1}{a} x_1^*$$



Engel curve for good 1

$$y = \frac{(a+b)p_2}{b} x_2^*$$



Engel curve for good 2

Income Changes and Perfectly-Complementary Preferences

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- Another example of computing the equations of Engel curves; the perfectly-complementary case.

$$U(x_1, x_2) = \min\{x_1, x_2\}.$$

- The ordinary demand equations are

$$x_1^* = x_2^* = \frac{y}{p_1 + p_2}.$$

Income Changes and Perfectly-Complementary Preferences

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$$x_1^* = x_2^* = \frac{y}{p_1 + p_2}.$$

Rearranged to isolate y, these are:

$$y = (p_1 + p_2)x_1^*$$ **Engel curve for good 1**

$$y = (p_1 + p_2)x_2^*$$ **Engel curve for good 2**

Income Changes

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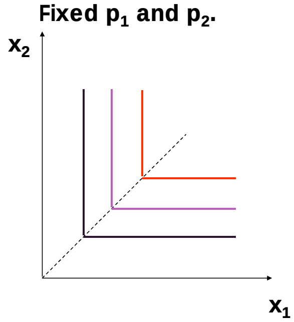
Fixed p_1 and p_2.



Income Changes

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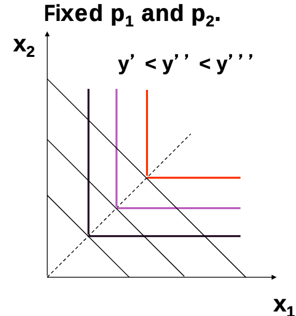

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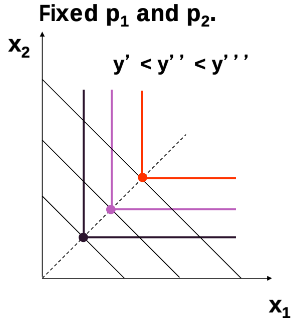

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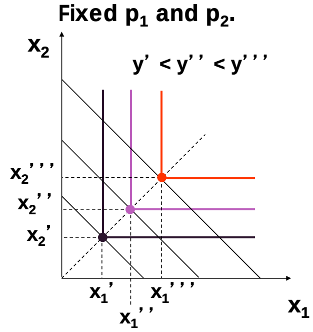

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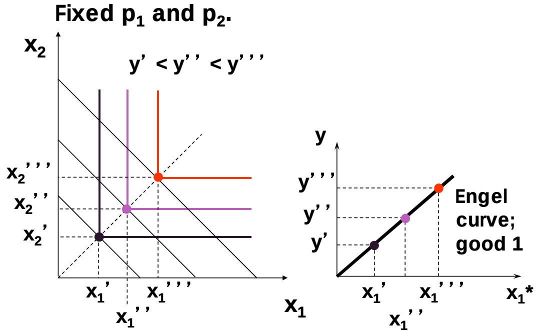

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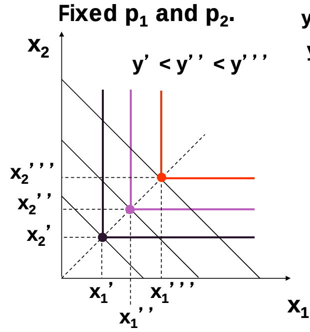

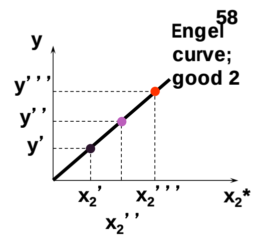

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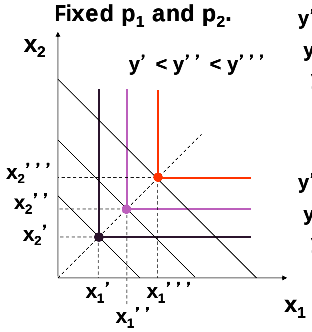

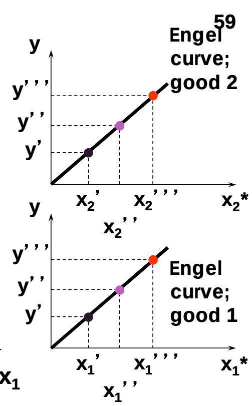

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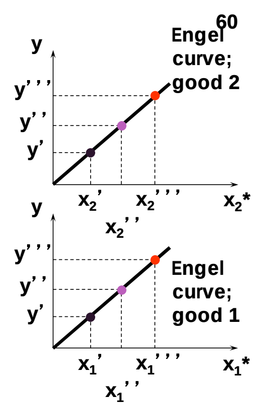
Income Changes

Fixed p_1 and p_2.

$$y = (p_1 + p_2)x_2^*$$

$$y = (p_1 + p_2)x_1^*$$



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Income Changes

- Some properties of the Engel curves that we have seen so far:
1. They are straight lines
2. They are upward sloping
- Is this true in general?

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Income Changes

- Some properties of the Engel curves that we have seen so far:
1. They are straight lines
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- Is this true in general?

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Homotheticity

- It turns out the linear Engel curves come from preferences which are **homothetic**
- A consumer's preferences are homothetic if and only if

$$(x_1,x_2) \prec (y_1,y_2) \Leftrightarrow (kx_1,kx_2) \prec (ky_1,ky_2)$$

for every k > 0.

- That is, the consumer's MRS is the **same anywhere on a straight line drawn from the origin.**
  - i.e. draw a straight line from the origin
  - The slope of indifference curves is the same everywhere along that line

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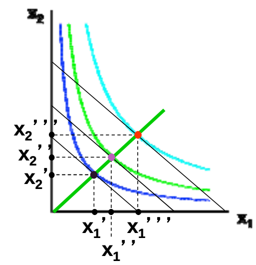
Income Changes



- MRS the same along a straight line
- As prices don't change, tangency point the same along a straight line
- Offer curve and Engle Curves are straight lines

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Income Effects -- A Nonhomothetic Example

- Are preferences always homothetic?
- No, in fact you have come across an example of on-homothetic preferences:
  - Quasilinear. For example:

$$U(x_1,x_2) = \sqrt{x_1} + x_2.$$

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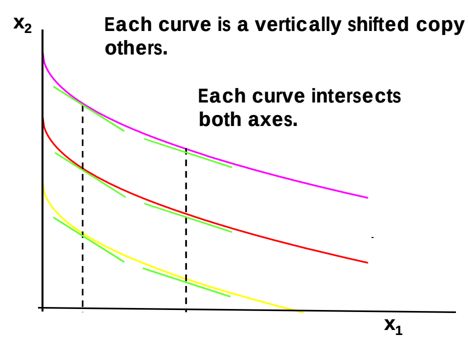
Quasi-linear Indifference Curves



Income Changes; Quasilinear Utility

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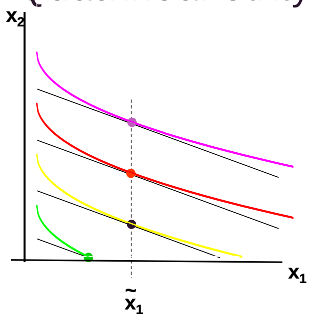

Income Changes; Quasilinear Utility

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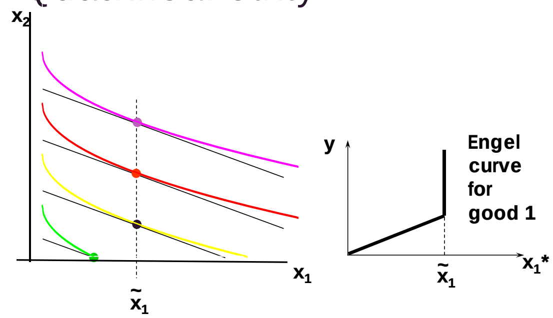
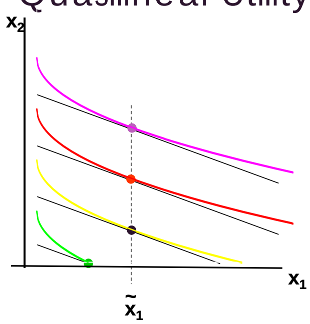
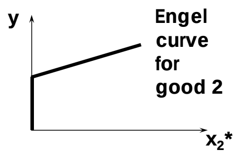


Income Changes; Quasilinear Utility

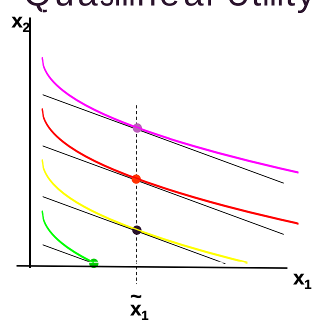
69

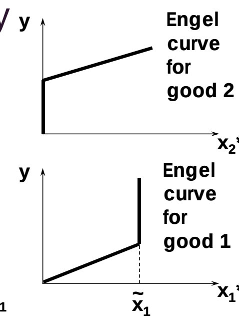




Income Changes; Quasilinear Utility

70





Income Changes

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- Some properties of the Engel curves that we have seen so far:

1. They are straight lines
2. They are upward sloping

- Is this true in general?

Income Effects

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- A good for which quantity demanded **rises** with income is called **normal.**
- Therefore a normal good's Engel curve is **positively sloped**.

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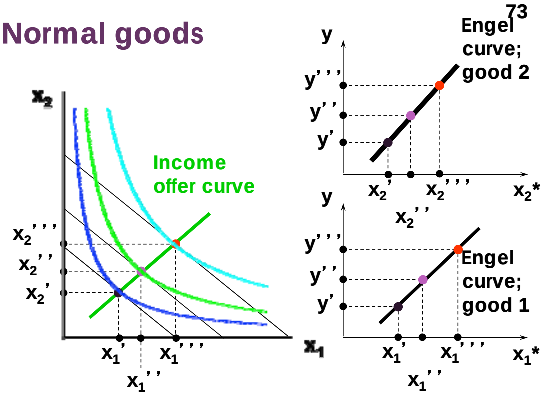
Normal goods



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Income Effects

- It is possible to construct examples in which this is not the case

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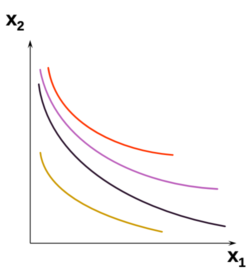
Income Effects: inferior goods



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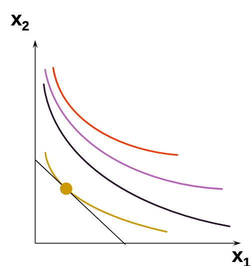
Income Effects



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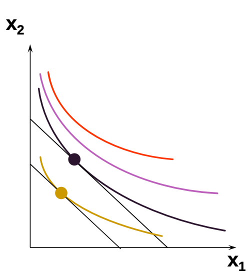
Income Effects



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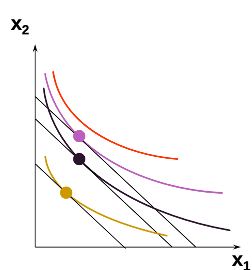
Income Effects



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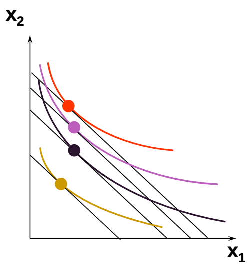
Income Effects



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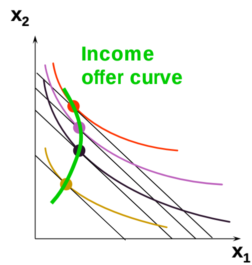
Income Effects



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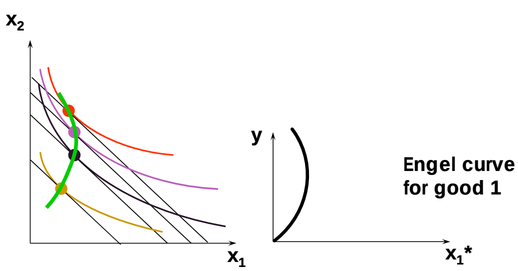
Income Effects



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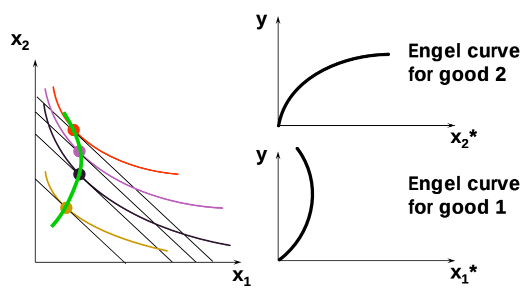
Income Effects



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Income Effects

- A good for which quantity demanded **falls** as income increases is called income **inferior**.
- Therefore an income inferior good's Engel curve is negatively sloped.

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Inferior Goods

- So good can be inferior goods
- But wait: why?
- Examples?

Inferior Goods

- So good can be inferior goods
- But wait: why?
- Examples?
  - Cheap objects: the wealthier you are the less you buy of them
  - Ramen noodles

Income Elasticity of Demand

- How do we mathematically describe the effect of income on demand?
- Derivative of the demand function?

$$\frac{\partial x_1(p_1, p_2, y)}{\partial y}$$

- Often we will instead use the **income elasticity of demand**

$$\frac{\partial x_1(p_1, p_2, y)}{\partial y}\frac{y}{x_1}$$

- We can thing of this as the effect of a percentage change in income on the percentage change in demand

$$\frac{\Delta x_1}{x_1} \Big/ \frac{\Delta y}{y}$$

Income Elasticity of Demand

- What is the advantage of using elasticities?
- One is that they do not depend on the units
- Imagine that we measure the demand for potatoes in kilos rather than pounds
  - The 'demand' at a given price would change
  - So would the derivation of the demand function
  - However, the elasticity would not change
  - A percentage change is the same whether demand is measured in kilos or pounds
- Equivalent concept of price elasticity of demand

$$\frac{\partial x_i(p_1, p_2, y)}{\partial p_j}\frac{p_j}{x_1}$$

Summary

Summary

- Today we have done the following

1. Provide some more mathematically sophisticated tools to find tangency points
   - Derivative based approach
   - Karush Kuhn Tucker Conditions
2. Defined the concept of 'demand', and shown how the demand for a particular good can change with income