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Intermediate Microeconomics W3211

Lecture 20: Game Theory 2

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Introduction

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The Story So Far….

- Last lecture we began to study strategic interactions
  - Situations in which your best action is affected by what I do, and vise versa
- This area of study is called game theory
- We defined a game
  - Players, actions, payoffs
- Talked about how to 'solve' a game
  - Iterated deletion of strictly dominated strategies
  - Nash equilibrium
- Showed that the Nash Equilibria
  - Aren't necessarily unique
  - Aren't necessarily efficient

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Today

- Talk about whether Nash Equilibrium always exist
  - If not, then there may not be much use
- Talk about what happens when games are played **sequentially**
  - i.e. first one player, then the other

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But First…..

- The beauty contest game…..

This week I will run the following contest: each of you can email me a number between 1 and 100. Whoever sends in the number that is closest to 2/3 of the average of all the numbers sent in gets the prize. What number should you send in?

- Can we solve this game using any of the techniques we used last time?
- Yes!
- Using the iterated deletion of strictly dominated strategies
- First, can it EVER be optimal to play 100?
- No. The highest the average can ever be is 100
- If this is the average, the best thing to do is play 66.666
- If the average is below 100, then the best thing to do will be to play something lower than this
- So 100 is strictly dominated
- In fact, everything over 66.66 is strictly dominated

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But First…..

- The beauty contest game…..

This week I will run the following contest: each of you can email me a number between 1 and 100. Whoever sends in the number that is closest to 2/3 of the average of all the numbers sent in gets the prize. What number should you send in?

- So we know that no one will ever play over 66.66
- Taking this into account, can it ever be optimal to play 66.66?
- No! The highest the average can be now is 66.66
- Which means that the highest you can ever want to play is 44.44
- Every number above 44.44 is now dominated, so we know that nothing over 44.44 will be played
- But can it be optimal to play 44.44?
- No!
- And so on and so forth
- Repeating this logic, the only strategy that survives is 0
- This is the prediction from the iterated deletion of strictly dominated strategies

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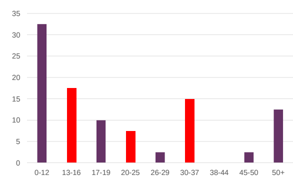
Your Data



- **Average: 22.07**
- **2/3 Average is 14.78**
- **Five Winners! Robert Gelinas, Sarina Perera, Danlei Wu, Xin Chang, Andy Kong**

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Failures of Equilibrium

- In the beauty contest game, very few of you played the action that survived the IESDS
- In fact, playing such a strategy would not have won you the game
- You people were not playing an equilibrium!
- Do we have any better models?

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Level K Thinking

- One popular model in the literature is the ‘Level k’ model
- It works as follows.
- First, imagine someone who played the game by picking a number at random
  - These are level 0 types
  - In the beauty contests game they would play 50 on average
- Now imagine someone who thinks about the game by assuming they are playing level 0 types, then pick the best thing to do
  - These are level 1 types
  - In the beauty contest game they would play 33.33
- Now think about someone who thinks they are playing level 1 types, and best respond to them
  - These are level 2 types
  - In the beauty contest game they would play 22.22
- And so on
- It is assumed that the world is a mixture of Level 1,2,3 etc types.

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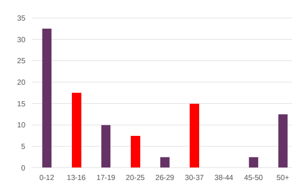
Your Data



- **Red bars indicate level 1, 2 and 3 play respectively**
- **Many level 1 and 3 types**
- **Also lots of rational types**
- **Well done!**

Nash Equilibrium:

Existence and Mixed Strategies

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Nash Equilibrium and Existence

- At least one reason why we want to use the idea of Nash Equilibrium is to make predictions
- We can only make predictions if the game has a Nash equilibrium
  - If not, we don’t have much to say....
- Do all games have a Nash Equilibrium?

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Pure Strategies

		Player B L	Player B R
Player A	U	(1,2)	(0,4)
Player A	D	(0,5)	(3,2)

Here is a new game. Are there any Nash equilibria?

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Pure Strategies

		Player B L	Player B R
Player A	U	(1,2)	(0,4)
Player A	D	(0,5)	(3,2)

Is (U,L) a Nash equilibrium? No.
Is (U,R) a Nash equilibrium? No.
Is (D,L) a Nash equilibrium? No.
Is (D,R) a Nash equilibrium? No.

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Pure Strategies

		Player B L	Player B R
Player A	U	(1,2)	(0,4)
Player A	D	(0,5)	(3,2)

So the game has no Nash equilibria in pure strategies. Even so, the game does have a Nash equilibrium, but in mixed strategies.

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Mixed Strategies

- Instead of playing purely Up or Down, Player A selects a probability distribution $(\pi_U, 1-\pi_U)$, meaning that with probability π_U Player A will play Up and with probability $1-\pi_U$ will play Down.
- Player A is mixing over the pure strategies Up and Down.
- The probability distribution $(\pi_U, 1-\pi_U)$ is a mixed strategy for Player A.

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Mixed Strategies

- Similarly, Player B selects a probability distribution $(\pi_L, 1-\pi_L)$, meaning that with probability π_L Player B will play Left and with probability $1-\pi_L$ will play Right.
- Player B is mixing over the pure strategies Left and Right.
- The probability distribution $(\pi_L, 1-\pi_L)$ is a mixed strategy for Player B.

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Mixed Strategies

		Player B L	Player B R
Player A	U	(1,2)	(0,4)
Player A	D	(0,5)	(3,2)

This game has no Nash equilibrium in pure strategies, but it does have a Nash equilibrium in mixed strategies. How is it computed?

Mixed Strategies

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		Player B	
		L, π_L	R, 1-π_R
Player A	U, π_U	(1,2)	(0,4)
	D, 1-π_U	(0,5)	(3,2)

Mixed Strategies

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		Player B	
		L, π_L	R, 1-π_L
Player A	U, π_U	(1,2)	(0,4)
	D, 1-π_U	(0,5)	(3,2)

A's expected (i.e. average) value of choosing Up is ??

Mixed Strategies

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		Player B	
		L, π_L	R, 1-π_L
Player A	U, π_U	(1,2)	(0,4)
	D, 1-π_U	(0,5)	(3,2)

A's expected value of choosing Up is π_L.
A's expected value of choosing Down is ??

Mixed Strategies

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		Player B	
		L, π_L	R, 1-π_L
Player A	U, π_U	(1,2)	(0,4)
	D, 1-π_U	(0,5)	(3,2)

A's expected value of choosing Up is π_L.
A's expected value of choosing Down is $3(1 - \pi_L)$.

Mixed Strategies

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		Player B	
		L, π_L	R, 1-π_L
Player A	U, π_U	(1,2)	(0,4)
	D, 1-π_U	(0,5)	(3,2)

A's expected value of choosing Up is π_L.
A's expected value of choosing Down is $3(1 - \pi_L)$.
If $\pi_L > 3(1 - \pi_L)$ then A will choose only Up, but there is no NE in which A plays only Up.

Mixed Strategies

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		Player B	
		L, π_L	R, 1-π_L
Player A	U, π_U	(1,2)	(0,4)
	D, 1-π_U	(0,5)	(3,2)

A's expected value of choosing Up is π_L.
A's expected value of choosing Down is $3(1 - \pi_L)$.
If $\pi_L < 3(1 - \pi_L)$ then A will choose only Down, but there is no NE in which A plays only Down.

Mixed Strategies

		Player B	
		L, π_L	R, $1-\pi_L$
Player A	U, π_U	(1,2)	(0,4)
	D, $1-\pi_U$	(0,5)	(3,2)

If there is a NE necessarily $\pi_L = 3(1 - \pi_L) \Rightarrow \pi_L = 3/4$; *i.e.* the way B mixes over Left and Right must make A indifferent between choosing Up or Down.

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Mixed Strategies

		Player B	
		L, 3/4	R, 1/4
Player A	U, π_U	(1,2)	(0,4)
	D, $1-\pi_U$	(0,5)	(3,2)

If there is a NE necessarily $\pi_L = 3(1 - \pi_L) \Rightarrow \pi_L = 3/4$; *i.e.* the way B mixes over Left and Right must make A indifferent between choosing Up or Down.

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Mixed Strategies

		Player B	
		L, 3/4	R, 1/4
Player A	U, π_U	(1,2)	(0,4)
	D, $1-\pi_U$	(0,5)	(3,2)

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Mixed Strategies

		Player B	
		L, 3/4	R, 1/4
Player A	U, π_U	(1,2)	(0,4)
	D, $1-\pi_U$	(0,5)	(3,2)

B's expected value of choosing Left is ??

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Mixed Strategies

		Player B	
		L, 3/4	R, 1/4
Player A	U, π_U	(1,2)	(0,4)
	D, $1-\pi_U$	(0,5)	(3,2)

B's expected value of choosing Left is $2\pi_U + 5(1 - \pi_U)$.
B's expected value of choosing Right is ??

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Mixed Strategies

		Player B	
		L, 3/4	R, 1/4
Player A	U, π_U	(1,2)	(0,4)
	D, $1-\pi_U$	(0,5)	(3,2)

B's expected value of choosing Left is $2\pi_U + 5(1 - \pi_U)$.
B's expected value of choosing Right is $4\pi_U + 2(1 - \pi_U)$.

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Mixed Strategies

		Player B	
		L, 3/4	R, 1/4
Player A	U, π_U	(1,2)	(0,4)
	D, $1-\pi_U$	(0,5)	(3,2)

B's expected value of choosing Left is $2\pi_U + 5(1 - \pi_U)$.
B's expected value of choosing Right is $4\pi_U + 2(1 - \pi_U)$.
If $2\pi_U + 5(1 - \pi_U) > 4\pi_U + 2(1 - \pi_U)$ then B will choose only Left, but there is no NE in which B plays only Left.

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Mixed Strategies

		Player B	
		L, 3/4	R, 1/4
Player A	U, π_U	(1,2)	(0,4)
	D, $1-\pi_U$	(0,5)	(3,2)

B's expected value of choosing Left is $2\pi_U + 5(1 - \pi_U)$.
B's expected value of choosing Right is $4\pi_U + 2(1 - \pi_U)$.
If $2\pi_U + 5(1 - \pi_U) < 4\pi_U + 2(1 - \pi_U)$ then B plays only Right, but there is no NE where B plays only Right.

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Mixed Strategies

		Player B	
		L, 3/4	R, 1/4
Player A	U, 3/5	(1,2)	(0,4)
	D, 2/5	(0,5)	(3,2)

If there is a NE then necessarily
$2\pi_U + 5(1 - \pi_U) = 4\pi_U + 2(1 - \pi_U) \Rightarrow \pi_U = 3/5$;
i.e. the way A mixes over Up and Down must make B indifferent between choosing Left or Right.

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Mixed Strategies

		Player B	
		L, 3/4	R, 1/4
Player A	U, 3/5	(1,2)	(0,4)
	D, 2/5	(0,5)	(3,2)

The game's only Nash equilibrium consists of A playing the mixed strategy (3/5, 2/5) and B playing the mixed strategy (3/4, 1/4).

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Mixed Strategies

		Player B	
		L, 3/4	R, 1/4
Player A	U, 3/5	(1,2) 9/20	(0,4)
	D, 2/5	(0,5)	(3,2)

The payoff will be (1,2) with probability
$3/5 \times 3/4 = 9/20$.

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Mixed Strategies

		Player B	
		L, 3/4	R, 1/4
Player A	U, 3/5	(1,2) 9/20	(0,4) 3/20
	D, 2/5	(0,5)	(3,2)

The payoff will be (0,4) with probability
$3/5 \times 1/4 = 3/20$.

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Mixed Strategies

Player B

Player A	L, 3/4	R, 1/4
U, 3/5	(1,2) 9/20	(0,4) 3/20
D, 2/5	(0,5) 6/20	(3,2)

The payoff will be (0,5) with probability
$2/5 \times 3/4 = 6/20.$

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Mixed Strategies

Player B

Player A	L, 3/4	R, 1/4
U, 3/5	(1,2) 9/20	(0,4) 3/20
D, 2/5	(0,5) 6/20	(3,2) 2/20

The payoff will be (3,2) with probability
$2/5 \times 1/4 = 2/20.$

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Mixed Strategies

Player B

Player A	L, 3/4	R, 1/4
U, 3/5	(1,2) 9/20	(0,4) 3/20
D, 2/5	(0,5) 6/20	(3,2) 2/20

A's NE expected payoff is
$1 \times 9/20 + 3 \times 2/20 = 3/4.$

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Mixed Strategies

Player B

Player A	L, 3/4	R, 1/4
U, 3/5	(1,2) 9/20	(0,4) 3/20
D, 2/5	(0,5) 6/20	(3,2) 2/20

A's NE expected payoff is
$1 \times 9/20 + 3 \times 2/20 = 3/4.$
B's NE expected payoff is
$2 \times 9/20 + 4 \times 3/20 + 5 \times 6/20 + 2 \times 2/20 = 16/5.$

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Mixed Strategies

- Why is this a Nash Equilibrium?
- Remember, A Nash Equilibrium means that I am taking my best action given what you are doing
- In this case, the actions of person B are such that person A gets the same payoff whatever they do
- So it is a best action for person A to randomize – in the sense that they cannot do better by taking any other action
- (Almost) all games have a Nash Equilibrium in mixed strategies

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Mixed Strategies

- But notice that something odd is going on.
- The actions of player B are set to make player A indifferent
- This means that it is the payoffs of player A that determine the actions of player B!
- Does this work in practice?

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Mixed Strategies

	L	R
U	80,40	40,80
D	40,80	80,40

- The following game was used in an experiment by Goree and Holt
- Is there a Nash Equilibrium in Pure Strategies?
- Equilibrium in mixed strategies requires utility of U to be the same as the utility of D for the row player

$$80\pi_L + 40(1-\pi_L) = 40\pi_L + 80(1-\pi_L)$$

- Implies $\pi_L = {}^1/_2$ and $\pi_U = {}^1/_2$

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Mixed Strategies

	L	R
U	80,40	40,80
D	40,80	80,40

- What do you think happened in the lab
- 48% of subjects played U and 48% played L
- Not bad!

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Mixed Strategies

	L	R
U	320,40	40,80
D	40,80	80,40

- What about this game?
- Equilibrium in mixed strategies requires utility of U to be the same as the utility of D for the row player

$$320\pi_L + 40(1-\pi_L) = 40\pi_L + 80(1-\pi_L)$$

- Implies $\pi_L = {}^1/_{16}$

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Mixed Strategies

	L	R
U	320,40	40,80
D	40,80	80,40

- Also requires utility of L to be the same as the utility of R for the column player

$$40\pi_U + 80(1-\pi_U) = 80\pi_U + 40(1-\pi_U)$$

- Implies $\pi_U = {}^1/_2$!
- So despite the fact that the payoff for (U,L) increased hugely for the row player, they still only play π_U half the time
- This is because they play in order to make the **column** player indifferent between L and R

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Mixed Strategies

	L	R
U	320,40	40,80
D	40,80	80,40

- What do you think happened in the data?
- Subjects played U 96% of the time and R 84% of the time
- You should check, but this is not a Nash Equilibrium

Sequential Games

Subgame Perfect Nash Equilibrium

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Sequential Games

- The type of games that we have looked at so far have been called **simultaneous move games**
- We have assumed that each player chooses their action at the same time
- Do not know what the other player has done when they choose their action
- Sometimes this is a good assumption (like the prisoner's dilemma)
- But not always...

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Sequential Games

- Think about a game between two firms
  - An entrant, who is deciding whether to get into the industry
  - An incumbent, who is already in the industry
- The choices of the two players are
  - Entrant has to choose whether to enter the industry or not
  - Incumbent has to decide whether to fight the entrant if they come in (with a price war)
- If the entrant stays out, the incumbent gets big profits (say 10) and the entrant gets zero
- If the entrant comes in and the incumbent fights, then both firms lose money (and get payoff -1)
- If the entrant comes in and the incumbent does not fight, then both sides get 5

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Sequential Games

	Enter	Stay Out
Fight	-1,-1	10,0
Relent	5,5	10,0

- What is the Nash Equilibrium of this game?
- There are two (Fight, Stay Out) and (Relent, Enter)
- Do we think that each of these is equally likely?
- Remember, the incumbent will decide whether to fight or not AFTER the entrant has decide whether to enter

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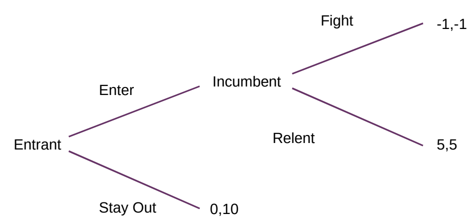
Sequential Games



- This is the **sequential form** of the game
- What would the Incumbent choose to do if the Entrant enters?

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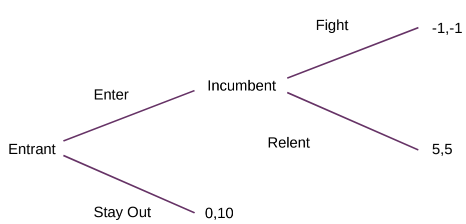
Sequential Games



- If they fight they get -1, if they relent they get 5
- So if they were forced to make that choice, then they would relent

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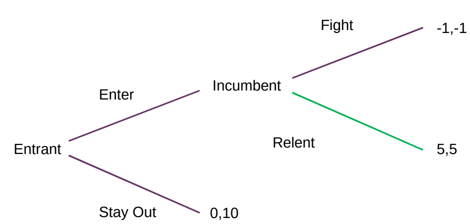
Sequential Games



- So what will the Entrant do?
- If they enter, they know that the Incumbent will relent and they will get 5
- If they stay out they will get 0

Sequential Games

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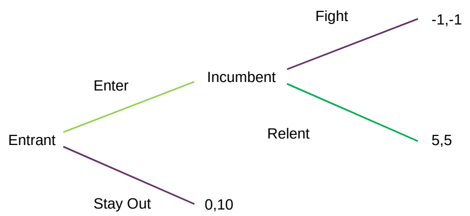

- So what will the Entrant do?
- If they enter, they know that the Incumbent will relent and they will get 5
- If they stay out they will get 0

Sequential Games

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- Of the two equilibria (Fight, Stay Out) or (Enter, Relent) only the latter seems plausible?
- Why?
- The Incumbent's threat to fight is not credible
- If it came to it, they would rather relent than fight
- The Entrant knows this
- So they come in to the industry

Subgame Perfect Nash Equilibrium

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- What we have just seen is an example of **Subgame Perfect Nash Equilibrium**
- We demand that the strategies of the players have to be an equilibrium, not only for the whole game, but also in each subgame
- What is a subgame?
- It is the rest of the game starting at any node (i.e. decision point
- The example we saw had two subgames

Sequential Games

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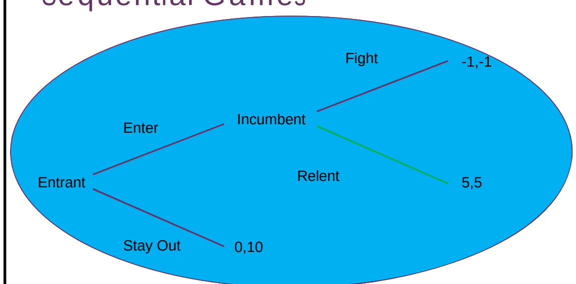

- The **WHOLE GAME** is a subgame

Sequential Games

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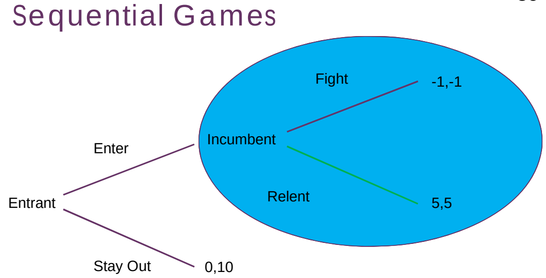

- There is also a subgame starting at the incumbent's choice

Sequential Games

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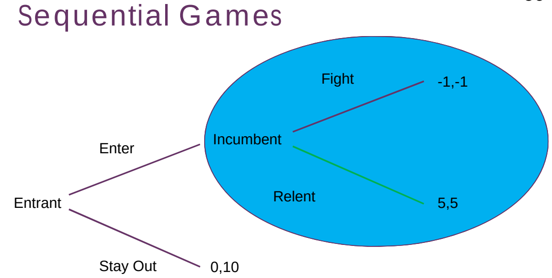

- The only equilibrium in this subgame is to play relent
- Thus, the only subgame perfect equilibrium is (Enter, Relent)

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Subgame Perfect Nash Equilibrium

- Subgame Perfect Nash Equilibrium sound very complicated
- But in fact they are very easy to find
- Simply start at the right hand side of the game
- Find the best action for those subgames
- Then find the best actions for the previous players, assuming that later players play their best action
- And so on....
- This is how we solved the game before
- It is called solving the game by **Backward Induction**

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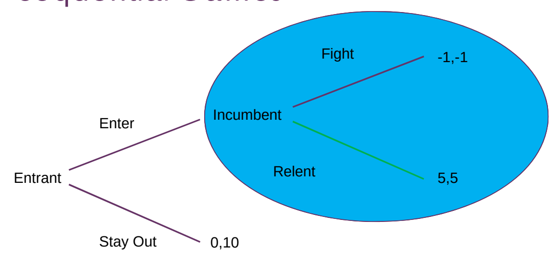
Sequential Games



- In this game we know that the Incumbent will play 'Relent'

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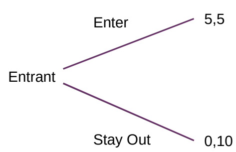
Sequential Games



- In this is the choice of the entrant, and they will pick Enter

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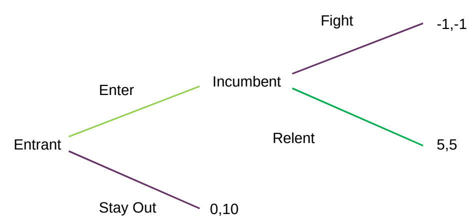
Sequential Games



- This is the subgame perfect nash equlibrium

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Sequential Games

- Another example: Bach or Stravinsky (previously know as the Battle of the Sexes game)
- Two people are trying to decide what to go and see: Bach or Stravinsky.
- Bert Prefers Bach, Sam prefers Stravinsky
- Both prefer seeing a concert together than seeing them apart

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Sequential Games

Sam

Bert

	Bach	Stravinsky
Bach	2,1	0,0
Stravinsky	0,0	1,2

- There are two Nash Equilibrium of this game
  - (B,B) and (S,S)
- But what if we make it a sequential game?
- Bert gets to choose first
- A strategy of Sam now has to tell us what they will do if Bert chooses Bach AND what they will do if Bert chooses Stravinsky

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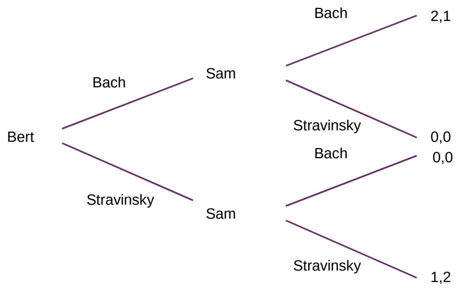
Bach and Stravinsky



- We can solve this game by backward induction
- What would Sam do in Bert chose Bach?

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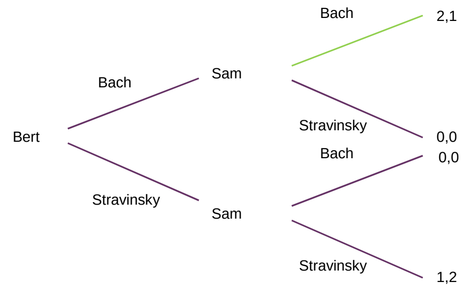
Bach and Stravinsky



- We can solve this game by backward induction
- What would Sam do if Bert chose Bach?

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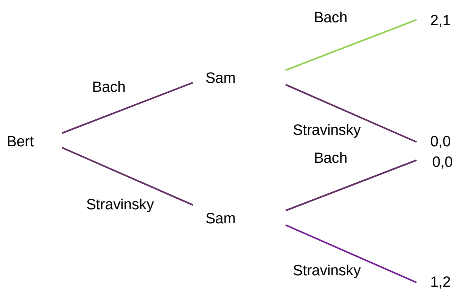
Bach and Stravinsky



- What would Sam do if Bert chose Stravinsky?

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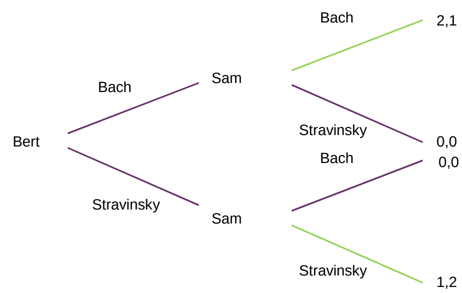
Bach and Stravinsky



- What would Sam do if Bert chose Stravinsky?

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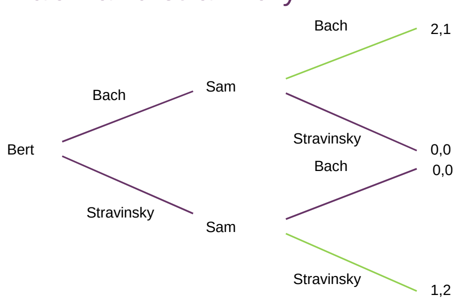
Bach and Stravinsky



- What should Bert do, given this is what Sam will do

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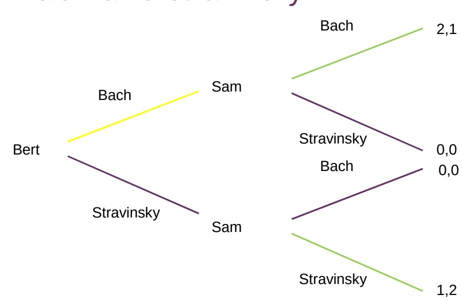
Bach and Stravinsky



- This is the SPNE of this game

Summary

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Today

- We have
- Talked about whether Nash Equilibrium always exist
  - Shown that they do if we have mixed strategies
- Talk about what happens when games are played **sequentially**
  - i.e. first one player, then the other

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Game Theory is Fun!

- Examples of questions you might be able to answer:

This story involves a village high up in the Italian Alps. The occupants of this village confirm to all currently available stereotypes. First, the men are lotharios, in the sense that some of them are cheating on their wives with the wives of other men. Second, they are dreadful gossips, so every man in the village knows whether every other man in the village is being cheated on by his wife (but he does not know about his own wife). Third, they are fiercely proud (and sexist hypocrites) - and each man declares that if he catches his own wife cheating, he will shoot her in the town square at midnight. Fourth, they are very religious, and all attend mass every Sunday. One Sunday, a new young firebrand priest turns up to give a sermon. As part of his sermon he condemns the town as a den of wickedness, with the words "everywhere I look in this village, I see sin. I know for a fact that some of the men in this village are lying with the wives of other men".

For the first night after the preacher leaves, all is quiet, as is the second night. On the third night, shots are heard in the square at midnight. The question is, how many shots were fired, and how many husbands were cheating on their wives