Intermediate Microeconomics W3211

Lecture 13: Perfect Competition 3: Solving the Cost Minimization Problem

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Introduction

The Story So Far….

- We have set up the firm's problem
  - Converts inputs to outputs in order to maximize profit
- Thought about how to solve the firm's problem when there is only one input
  - E.g. economists who only need labor to produce paper
- Started to think about how to solve the problem when the firm needs to use more than one input to produce output
- Suggested that we can solve this problem by splitting it into two
  1. Construct the **cost function** for the firm, by finding the lowest cost way of producing each output (the cost minimization problem)
  2. Choose the output level that maximizes profit given these costs (the profit maximization problem)
- Started to think about how to solve the cost minimization problem

Today

- Think more about how to solve the cost minimization problem
- Discuss Returns to Scale in the case of multiple inputs

The Cost Minimization Problem

The Case of Multiple Inputs

- Remember, last time we set up the firm's **cost minimization** problem

1. **CHOOSE** **inputs** $x_1, x_2, \ldots$
2. **IN ORDER TO MINIMIZE** **costs** $p_1x_1 + p_2x_2 + \cdots$
3. **SUBJECT TO** **achieving output** $y = f(x_1, x_2, x_3, \ldots.)$

- This will tell us the cost associated with each level of output

$$c(p_1, p_2, \ldots y)$$

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The Cost Minimization Problem

- Graphically, we said that we can solve this problem using two lines
- First is the 'iso cost' line, which links together pairs of inputs that have the same cost
- Costs are given by
$$c = p_1x_1 + p_2x_2$$
- So an iso cost line is
$$x_2 = \frac{c}{p_2} - \frac{p_1}{p_2}x_1$$

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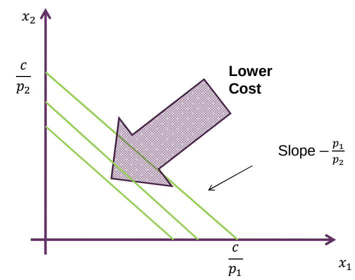
Iso-Cost Line



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The Cost Minimization Problem

- Second, we needed the **iso-output** lines
  - Find all the x_1 and x_2 such that $f(x_1, x_2) = y$ for some y
  - These are basically the same as indifference curves

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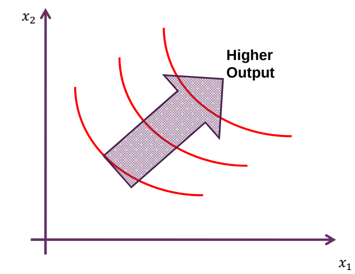
Iso-Output Lines



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The Cost Minimization Problem

- So now we can represent the firm's cost minimization problem in pictures
- A given output level y fixes an iso-output line
- We want to get on the left most (lowest) iso-cost line while staying on this iso-output line

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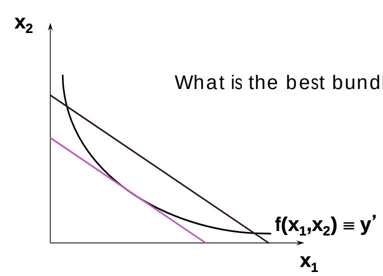
The Cost-Minimization Problem



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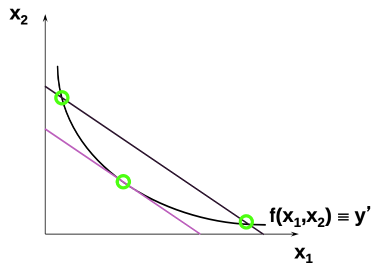
The Cost-Minimization Problem



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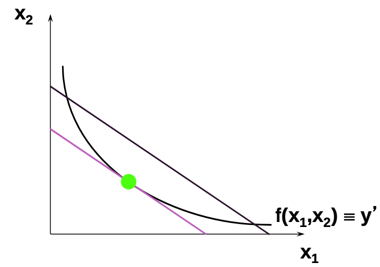
The Cost-Minimization Problem



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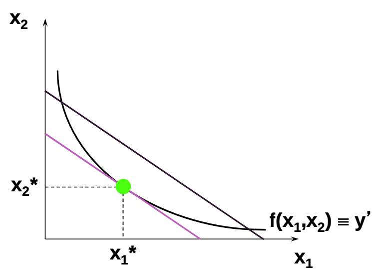
The Cost-Minimization Problem



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The Cost Minimization Problem

- Luckily, because this is so similar to the consumer's problem, you are automatically expert at solving this problem as well!
- There are three possible types of solution
  - Points of tangency
  - Kinks
  - Corner solutions
- The same recipe that worked for the consumer problem will also work here
  - Find all the points of tangency, kinks, and corner solutions
  - Find the cheapest out of all these possibilities
- And the same words of warning apply here!
  - Not all solutions are tangency points
  - Not all tangency points are solutions!

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Tangency Points

- Let's think for a minute about solutions that are tangency points

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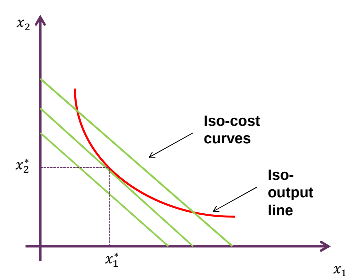
Tangency Solutions



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Tangency Points

- Let's think for a minute about solutions that are tangency points
- These are the points at which the following two things are equal
  - The slope of the iso-cost line
  - The slope of the iso-output line
- What are these two slopes?
- Remember, the iso-cost line is $x_2 = \frac{c}{p_2} - \frac{p_1}{p_2}x_1$
- The slope is therefore $\frac{p_1}{p_2}$ - the relative price of the two inputs.

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Tangency Points

- What about the slope of the iso-output line?
- This is the rate at which you can exchange one input for another **keeping output constant**
- We call this the **Marginal Rate of Technical Substitution (MRTS)**
- (Actually, as with the MRS for consumers, we will call the MRTS the NEGATIVE of the slope of the iso-output line)

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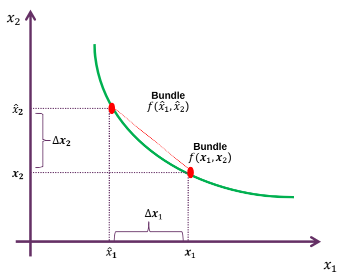
The Marginal Rate of Technical Substitution



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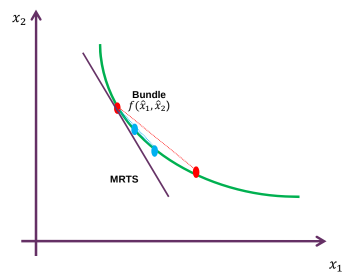
The Marginal Rate of Technical Substitution



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Tangency Points

- So the tangency point is the point at which the MRTS is equal to the relative prices of the two inputs
- As with the consumer's problem this makes sense
- Imagine that we weren't at a point of tangency

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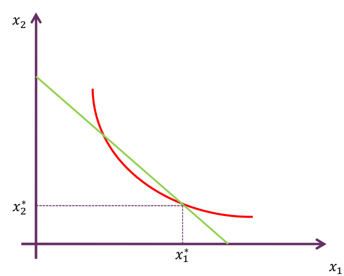
Tangency Solutions



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Tangency Points

- At this point the iso cost line is **steeper** that the iso-output line
- Imagine I reduce my use of good 1 and increase my use of good 2 to keep output constant

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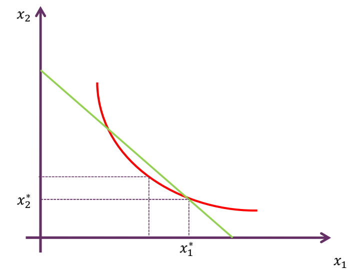
Tangency Solutions



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Tangency Points

- At this point the iso cost line is **steeper** that the iso-output line
- Imagine I reduce my use of good 1 and increase my use of good 2 to keep output constant
- The additional amount of good 2 I need to use to keep output constant is **less** than the price of good 2 relative to good 1

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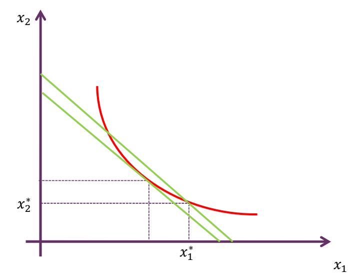
Tangency Solutions



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Tangency Points

- At this point the iso cost line is **steeper** that the iso-output line
- Imagine I reduce my use of good 1 and increase my use of good 2 to keep output constant
- The additional amount of good 2 I need to use to keep output constant is **less** than the price of good 2 relative to good 1
- This will therefore reduce costs
- I can always do this **unless** I am at a point of tangency

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Tangency Points

- To solve the cost minimization I need to be able to calculate
1. The slope of the iso-cost line
2. The slope of the iso output line
- 1 is easy – its just the ratio of the prices
- 2 is the MRTS
- How do I calculate the MRTS?

Calculating MRST

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- How do we calculate the Marginal Rate of Technical Substitution?
- Same trick as we used in the consumer's problem.
- For an iso output line we have

$$f(x_1, x_2) = y$$

- Take the total derivative with respect to x_1 *and* $, x_2$

$$\frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 = 0$$

- Which implies

$$\text{MRTS} = -\frac{dx_2}{dx_1} = \frac{\partial f}{\partial x_1} \Big/ \frac{\partial f}{\partial x_2}$$

Calculating MRST

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$$\text{MRTS} = \frac{\partial f}{\partial x_1} \Big/ \frac{\partial f}{\partial x_2}$$

- $\frac{\partial f}{\partial x_1}$ is the **marginal product** with respect to good 1
- It is the amount of additional output one would get using one additional unit of good 1
  - Keeping fixed the amount of good 2
- Thus the Marginal Rate of Technical Substitution is equal to the ratio of the marginal product with respect to each good

A Worked Example

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- Let's say that technology is given by

$$f(x_1, x_2) = x_1^{\frac{1}{3}} x_2^{\frac{1}{2}}$$

- Because this is Cobb Douglas technology, we know that we have an tangency point solution (right?)
- Taking derivatives we get

$$\frac{\partial f}{\partial x_1} = \frac{1}{3} x_1^{-\frac{2}{3}} x_2^{\frac{1}{2}}$$

$$\frac{\partial f}{\partial x_2} = \frac{1}{2} x_1^{\frac{1}{3}} x_2^{-\frac{1}{2}}$$

$$MRTS = \frac{2}{3}\frac{x_2}{x_1}$$

A Worked Example

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- The tangency condition therefore gives us

$$\frac{p_1}{p_2} = \frac{2}{3}\frac{x_2}{x_1}$$

Or $x_1 = \frac{2p_2}{3p_1} x_2$

- In order to create an amount of output y it must be the case that

$$x_1^{\frac{1}{3}} x_2^{\frac{1}{2}} = y$$

Or $\left(\frac{2p_2}{3p_1} x_2\right)^{\frac{1}{3}} x_2^{\frac{1}{2}} = y$

Which implies $x_2 = \left(\frac{3p_1}{2p_2}\right)^{\frac{2}{5}} y^{\frac{6}{5}}$ and $x_1 = \left(\frac{2p_2}{3p_1}\right)^{\frac{3}{5}} y^{\frac{6}{5}}$

A Worked Example

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$x_2^*(y) = \left(\frac{3p_1}{2p_2}\right)^{\frac{2}{5}} y^{\frac{6}{5}}$ and $x_1^*(y) = \left(\frac{2p_2}{3p_1}\right)^{\frac{3}{5}} y^{\frac{6}{5}}$

- So we now know how much of each input the firm will use in order to generate output y
- How do we turn this into a cost function?
- Well, the cost of using inputs x_1 and x_2 is

$$c = p_1 x_1 + p_2 x_2$$

- And so, substituting in we get

$$c(p_1, p_2, y) = \left(p_1^{\frac{2}{5}} p_2^{\frac{3}{5}} \left[\left(\frac{2}{3}\right)^{\frac{3}{5}} + \left(\frac{3}{2}\right)^{\frac{2}{5}}\right]\right) y^{\frac{6}{5}}$$

Returns to Scale and Cost Functions

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Returns to Scale and Cost Functions

- The reason we have calculated the cost function is so we can now solve the firm's profit maximization problem

1. **CHOOSE** y
2. **IN ORDER TO MAXIMIZE** $p_y y - c(p_1, y)$

- What the solution to this problem looks like will depend on what the cost function looks like
- What the cost function will look like will depend on the technology of the firm – i.e. the production function

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Returns to Scale and Cost Functions

- Lets think back to the case of one input
- Production function $f(x_1)$
- We made the assumption that $\frac{d^2 f}{dx_1^2} < 0$
  - i.e. marginal product decreased as we used more of the input
- This is equivalent to assuming **diminishing returns to scale**
- If you double inputs you don't double output
  - $f(kx_1) < kf(x_1)$

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Returns to Scale and Cost Functions

- Of course, you could think of different types of cost functions:
- **Decreasing** returns to scale: doubling the inputs **less** than doubles the output

$$kf(x_1, x_2, x_3, \ldots) > f(kx_1, kx_2, kx_3, \ldots)$$

- **Constant** returns to scale: doubling the inputs **exactly** doubles the output

$$kf(x_1, x_2, x_3, \ldots) = f(kx_1, kx_2, kx_3, \ldots)$$

- **Increasing** returns to scale: doubling the inputs **more** than doubles the output

$$kf(x_1, x_2, x_3, \ldots) < f(kx_1, kx_2, kx_3, \ldots)$$

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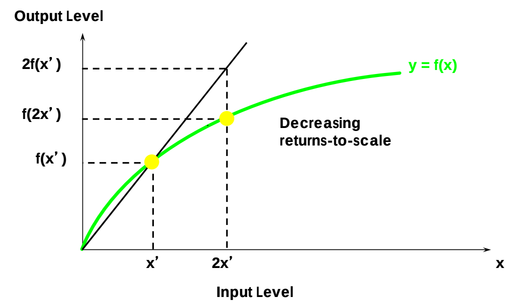
Returns-to-Scale

One input, one output



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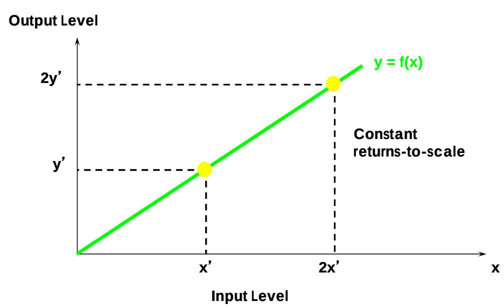
Returns-to-Scale

One input, one output



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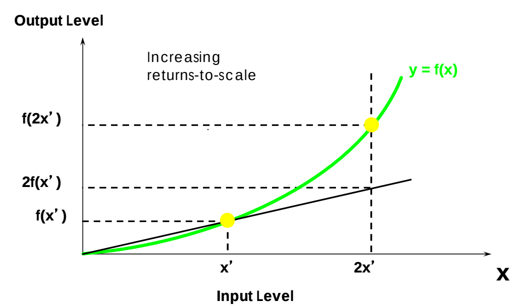
Returns-to-Scale

One input, one output



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Returns-to-Scale

- A single technology can 'locally' exhibit different returns-to-scale.

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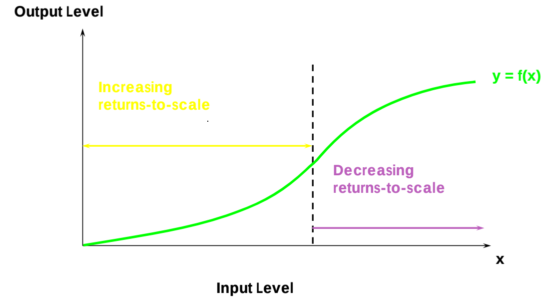
Returns-to-Scale

One input, one output



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Returns to Scale and Cost Functions

- What about the case where there are multiple inputs?
- Let's go back to Cobb Douglas technology
  $f(x_1, x_2) = x_1^{a_1} x_2^{a_2}$
- Is this increasing, decreasing or constant returns to scale?
- What happens when we increase the inputs by k
  $f(kx_1, kx_2) = (kx_1)^{a_1}(kx_2)^{a_2}$
  $=(k^{a_1+a_2})x_1^{a_1}x_2^{a_2}$
  $=(k^{a_1+a_2})f(x_1, x_2)$
- Is this increasing, decreasing or constant returns to scale?

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Returns to Scale and Cost Functions

$(k^{a_1+a_2})f(x_1, x_2)$

- Is this increasing, decreasing or constant returns to scale?
- Depends on $a_1 + a_2$!
- A Cobb Douglas production function
  $f(x_1, x_2, x_3, \ldots) = x_1^{a_1} x_2^{a_2} x_3^{a_3} \ldots$

Will exhibit

- **Decreasing** returns to scale if $a_1 + a_2 + a_3 \ldots < 1$
- **Constant** returns to scale if $a_1 + a_2 + a_3 \ldots = 1$
- **Increasing** returns to scale if $a_1 + a_2 + a_3 \ldots < 1$

Summary

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Summary

- Today we discussed how to solve the firm's cost minimization problem
- Discuss Returns to Scale in the case of multiple inputs