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Intermediate Microeconomics W3211

Lecture 14:
Cost Functions and Optimal Output

Columbia University, Spring 2016
Mark Dean: mark.dean@columbia.edu

Introduction

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The Story So Far….

- We have set up the firm's problem
  - Converts inputs to outputs in order to maximize profit
- Thought about how to solve the firm's problem when there is only one input
  - E.g. economists who only need labor to produce paper
- Thought about how to solve the problem when the firm needs to use more than one input to produce output
- Suggested that we can solve this problem by splitting it into two
  1. Construct the **cost function** for the firm, by finding the lowest cost way of producing each output (the cost minimization problem)
  2. Choose the output level that maximizes profit given these costs (the profit maximization problem)
- Figured out how to solve the firm's cost minimization problem

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Today

- Describe the relationship between returns to scale and cost functions
- Solve the second part of the firm's problem:
  - Choose the output level that maximizes profit given costs
  - i.e. the profit maximization problem
- Think about comparative statics
  - i.e. what happens when we change the prices of inputs and outputs

Returns to Scale and Cost Functions

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Returns to Scale and Cost Functions

- In the last lecture we defined **returns to scale** for production functions
- **Decreasing** returns to scale: doubling the inputs **less** than doubles the output

$$kf(x_1, x_2, x_3, \dots) > f(kx_1, kx_2, kx_3, \dots)$$

- **Constant** returns to scale: doubling the inputs **exactly** doubles the output

$$kf(x_1, x_2, x_3, \dots) = f(kx_1, kx_2, kx_3, \dots)$$

- **Increasing** returns to scale: doubling the inputs **more** than doubles the output

$$kf(x_1, x_2, x_3, \dots) < f(kx_1, kx_2, kx_3, \dots)$$

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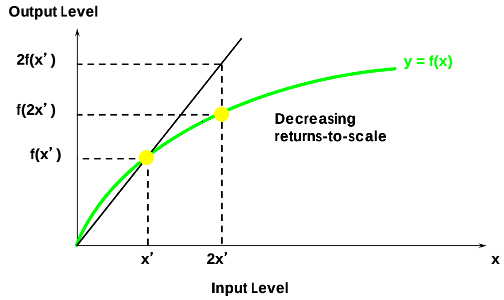
Returns-to-Scale

One input, one output



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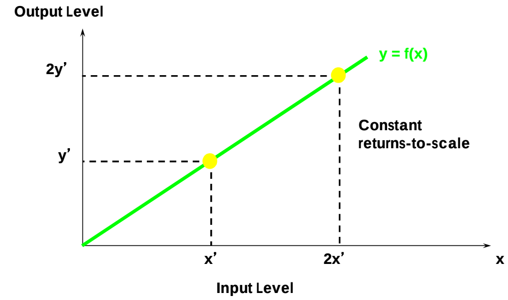
Returns-to-Scale

One input, one output



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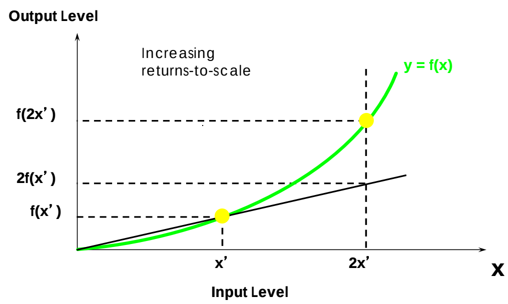
Returns-to-Scale

One input, one output



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Returns to Scale and Cost Functions

- We showed that, a Cobb Douglas production function
$$f(x_1, x_2, x_3, \ldots) = x_1^{a_1} x_2^{a_2} x_3^{a_3} \ldots$$

Will exhibit
- **Decreasing** returns to scale if $a_1 + a_2 + a_3 \ldots < 1$
- **Constant** returns to scale if $a_1 + a_2 + a_3 \ldots = 1$
- **Increasing** returns to scale if $a_1 + a_2 + a_3 \ldots > 1$

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Returns to Scale and Cost Functions

- Ultimately, what we are interested in is the firm's **costs**
- Specifically, their **marginal costs**, which will determine optimal output
- We showed that, in the case of one input, that there was a relationship between returns to scale and marginal costs
- Because of the inverse function theorem

$$\frac{\partial c(p_1, y)}{\partial y} = p_1 \frac{\partial f^{-1}(y)}{\partial y} = p_1 \frac{1}{\partial f(x_1) / \partial x_1}$$

- E.g., decreasing returns to scale means increasing marginal costs

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Returns to Scale and Cost Functions

- What are the marginal costs for a firm with Cobb Douglas production function?
- In the worked example, we showed that for technology

$$f(x_1, x_2) = x_1^{\frac{1}{3}} x_2^{\frac{1}{2}}$$

- Costs were given by

$$c(p_1, p_2, y) = \left(p_1^{\frac{2}{5}} p_2^{\frac{3}{5}} \left[\left(\frac{2}{3}\right)^{\frac{3}{5}} + \left(\frac{3}{2}\right)^{\frac{2}{5}}\right]\right) y^{\frac{6}{5}}$$

- It turns out (and you should check) that for technology

$$x_1^{a_1} x_2^{a_2}$$

- Costs are given by

$$c(p_1, p_2, y) = \left(p_1^{\frac{a_1}{a_1+a_2}} p_2^{\frac{a_2}{a_1+a_2}} A\right) y^{\frac{1}{a_1+a_2}}$$

- Where A is a constant which depends on a_1 and a_2

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Returns to Scale and Cost Functions

$$c(p_1, p_2, y) = \left(p_1^{\frac{a_1}{a_1+a_2}} p_2^{\frac{a_2}{a_1+a_2}} A\right) y^{\frac{1}{a_1+a_2}}$$

- What does this mean for marginal costs?
- Take the first derivative with respect to y

$$\frac{\partial c(p_1,p_2,y)}{\partial y} = \frac{1}{a_1+a_2}\left(p_1^{\frac{a_1}{a_1+a_2}} p_2^{\frac{a_2}{a_1+a_2}} A\right) y^{\frac{1}{a_1+a_2}-1}$$

- Now take the second derivative

$$\frac{\partial^2 c(p_1,p_2,y)}{\partial y^2} = \frac{1}{a_1+a_2}\left(\frac{1}{a_1+a_2}-1\right)\left(p_1^{\frac{a_1}{a_1+a_2}} p_2^{\frac{a_2}{a_1+a_2}} A\right) y^{\frac{1}{a_1+a_2}-2}$$

- Is this positive or negative?
  - i.e. are marginal costs increasing or decreasing?

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Returns to Scale and Cost Functions

$$\frac{\partial^2 c(p_1,p_2,y)}{\partial y^2} = \frac{1}{a_1+a_2}\left(\frac{1}{a_1+a_2}-1\right)\left(p_1^{\frac{a_1}{a_1+a_2}} p_2^{\frac{a_2}{a_1+a_2}} A\right) y^{\frac{1}{a_1+a_2}-2}$$

- Is this positive?
- Only if $\frac{1}{a_1+a_2}$ is greater than 1
  - i.e. only if $a_1 + a_2$<1
- i.e. Only if the production function exhibits decreasing returns to scale

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Returns to Scale and Cost Functions

- So, if there is only one input, or technology is Cobb Douglas
- **Decreasing** returns to scale if and only if marginal costs **increase** as y increases
- **Constant** returns to scale if and only if marginal cost **unchanged** as y increases
- **Increasing** returns to scale: marginal cost **fall** as y increases

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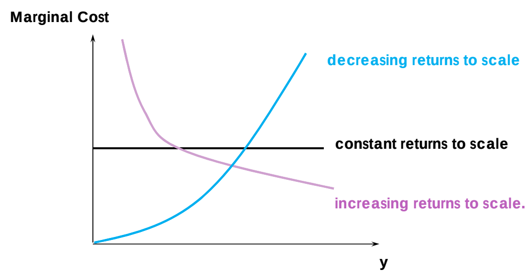
Returns-to-Scale and Marginal Cost



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Returns to Scale and Cost Functions

- What about Average Cost?
- Say marginal cost is always **increasing**
- The cost of the making the second car is higher than the cost of making the first car
- The **average** cost of producing two cars is higher than that of producing one car
- Following the same logic
  - **Decreasing** returns to scale → **increasing** marginal cost → **increasing** average cost
  - **Constant** returns to scale → **constant** marginal cost → **constant** average cost
  - **Increasing** returns to scale → **decreasing** marginal cost → **decreasing** average cost

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Returns-to-Scale and Total Costs

- What does this imply for the shapes of total cost functions?
- Lets start by thinking about **decreasing** returns to scale
- And therefore **increasing** average cost

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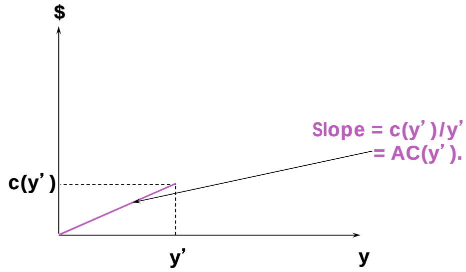
Returns-to-Scale and Total Costs



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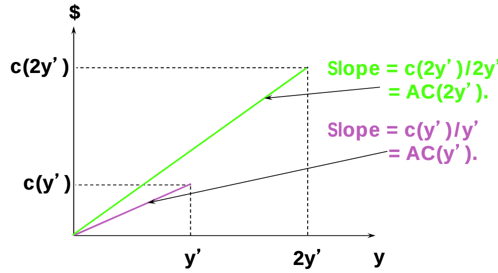
Returns-to-Scale and Total Costs



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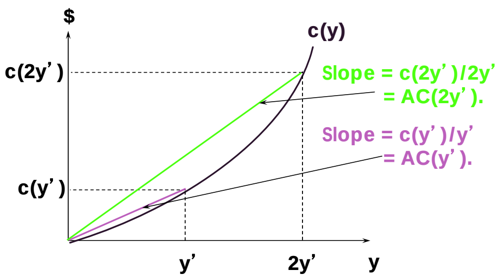
Returns-to-Scale and Total Costs



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+ Returns-to-Scale and Total Costs

- What does this imply for the shapes of total cost functions?
- Now **increasing** returns to scale
- And therefore **decreasing** average cost

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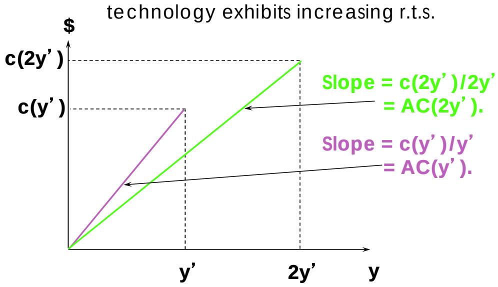
Returns-to-Scale and Total Costs

Av. cost decreases with y if the firm's technology exhibits increasing r.t.s.



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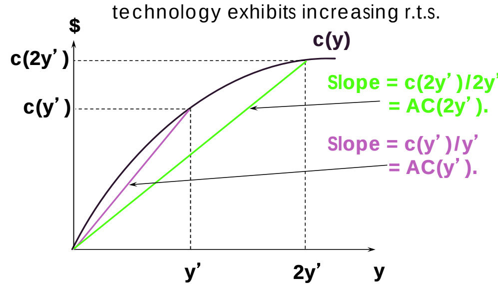
Returns-to-Scale and Total Costs

Av. cost decreases with y if the firm's technology exhibits increasing r.t.s.



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Returns-to-Scale and Total Costs

- What does this imply for the shapes of total cost functions?
- Finally **constant** returns to scale
- And therefore **constant** average cost

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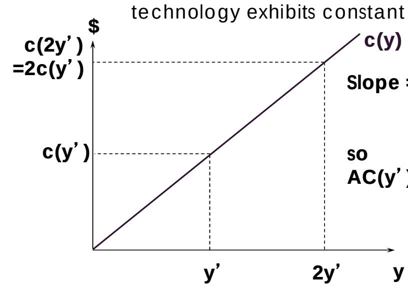
Returns-to-Scale and Total Costs

Av. cost is constant when the firm's technology exhibits constant r.t.s.



Slope = c(2y')/2y'
= 2c(y')/2y'
= c(y')/y'
so
AC(y') = AC(2y').

Solving The Firm's Profit Maximization Problem

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Profit Maximization

- Remember that the firm's original problem was

1. **CHOOSE inputs and output -** $y, x_1, x_2, \ldots$
2. **IN ORDER TO MAXIMIZE profit -** $p_y y - p_1 x_1 - p_2 x_2 - p_3 x_3 - \cdots$
3. **SUBJECT TO technological constraints -** $y \leq f(x_1, x_2, x_3, \ldots .)$

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Profit Maximization

- Which we split into two: First the **cost minimization** problem

1. **CHOOSE inputs** $x_1, x_2, \ldots$
2. **IN ORDER TO MINIMIZE costs** $p_1 x_1 + p_2 x_2 + \cdots$
3. **SUBJECT TO achieving output** $y = f(x_1, x_2, x_3, \ldots .)$

- This allowed us to identify the cost function $c(p_1, p_2, \ldots y)$
- Which we can then use to solve the **profit maximization** problem

1. **CHOOSE output** y
2. **IN ORDER TO MAXIMIZE profits** $p_y y - c(p_1, p_2, \ldots y)$

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Profit Maximization

- Let's think a bit more about the profit maximization problem
- Using the Cobb Douglas case as an example.
- Remember I claimed that for Cobb Douglas technology

$$f(x_1, x_2) = x_1^{a_1} x_2^{a_2}$$

- The cost function is given by

$$c(p_1, p_2, y) = \left(p_1^{\frac{a_1}{a_1+a_2}} p_2^{\frac{a_2}{a_1+a_2}} A\right) y^{\frac{1}{a_1+a_2}}$$

- And so profits by

$$p_y y - \left(p_1^{\frac{a_1}{a_1+a_2}} p_2^{\frac{a_2}{a_1+a_2}} A\right) y^{\frac{1}{a_1+a_2}}$$

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Profit Maximization

$$p_y y - \left(p_1^{\frac{a_1}{a_1+a_2}} p_2^{\frac{a_2}{a_1+a_2}} A\right) y^{\frac{1}{a_1+a_2}}$$

- How can we find the profit maximizing output?
- Well, we could take derivatives with respect to y
- And use the first order conditions

$$p_y = \frac{\partial c(p_1, y)}{\partial y} = MC(y)$$

$$p_y = \frac{1}{a_1 + a_2}\left(p_1^{\frac{a_1}{a_1+a_2}} p_2^{\frac{a_2}{a_1+a_2}} A\right) y^{\frac{1}{a_1+a_2}-1}$$

- Which implies

$$y = \left[\frac{p_y(a_1 + a_2)}{p_1^{\frac{a_1}{a_1+a_2}} p_2^{\frac{a_2}{a_1+a_2}} A}\right]^{\frac{a_1+a_2}{1-a_1-a_2}}$$

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Profit Maximization

- Is this going to work?
- Depends on whether we have **increasing**, **decreasing** or **constant** returns to scale
- If we have decreasing returns to scale (i.e. $a_1 + a_2$<1) then marginal costs are **increasing**

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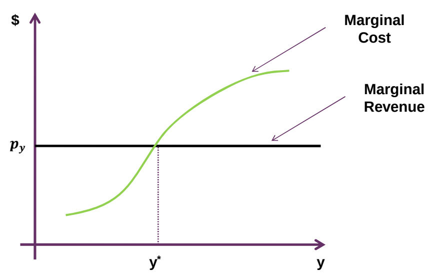
Marginal Revenue and Marginal Cost



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Profit Maximization

- In this case, setting marginal cost equal to price is optimal
- The cost of producing one more unit of output would be higher than the revenue of doing so
- This would lower profits
- We can also see this from the second order conditions

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Profit Maximization

- Profits are given by

$$p_y y - c(p_1, p_2, \ldots y)$$

$$p_y y - \left(p_1^{\frac{a_1}{a_1+a_2}} p_2^{\frac{a_2}{a_1+a_2}} A\right) y^{\frac{1}{a_1+a_2}}$$

- First derivative is

$$p_y - MC(y)$$

$$p_y - \frac{1}{a_1 + a_2}\left(p_1^{\frac{a_1}{a_1+a_2}} p_2^{\frac{a_2}{a_1+a_2}} A\right) y^{\frac{1}{a_1+a_2}-1}$$

- Second derivative is

$$-\frac{dMC(y)}{dy}$$

$$-\frac{1}{a_1 + a_2}\left(\frac{1}{a_1 + a_2} - 1\right)\left(p_1^{\frac{a_1}{a_1+a_2}} p_2^{\frac{a_2}{a_1+a_2}} A\right) y^{\frac{1}{a_1+a_2}-2}$$

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Profit Maximization

$$-\frac{1}{a_1 + a_2}\left(\frac{1}{a_1 + a_2} - 1\right)\left(p_1^{\frac{a_1}{a_1+a_2}} p_2^{\frac{a_2}{a_1+a_2}} A\right) y^{\frac{1}{a_1+a_2}-2}$$

- Remember, from calculus, for us to have found a maximum, we need this to be **negative**
- Which it will be if $\left(\frac{1}{a_1+a_2} - 1\right)$ is positive
- i.e. $a_1 + a_2$<1
- i.e. decreasing returns to scale

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Profit Maximization

- What about if we have increasing returns to scale?
- Then marginal costs are **decreasing**

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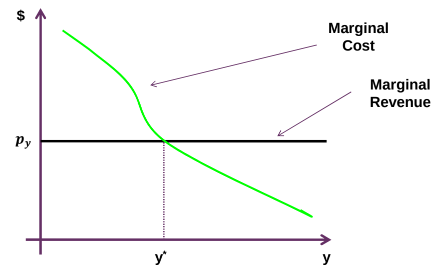
Marginal Revenue and Marginal Cost



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Profit Maximization

- Imagine we are producing at y^*
- What would happen if we produced one more unit of output?
- Now the costs of additional output are **below** revenue
- If we wanted to produce at y^* then we would certainly want to produce more
- The more we produce, the lower the costs and the higher the profits
- y^* is the **profit minimizing** output
- You can see this from the second order conditions

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Profit Maximization

$$-\frac{1}{a_1+a_2}\left(\frac{1}{a_1+a_2}-1\right)\left(p_1^{\frac{a_1}{a_1+a_2}}p_2^{\frac{a_2}{a_1+a_2}}A\right)y^{\frac{1}{a_1+a_2}-2}$$

- If $a_1+a_2>1$ (i.e. increasing returns to scale) then this expression is **positive**
- Means we have found a **profit minimizing** level of output
- So, if the 'tangency point' is not the solution, what is?
- A corner solution!
- Produce infinite output (assuming marginal costs at some point get below price)

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Profit Maximization

- What about constant returns to scale?
- $a_1+a_2=1$

$$MC(y)=\frac{1}{a_1+a_2}\left(p_1^{\frac{a_1}{a_1+a_2}}p_2^{\frac{a_2}{a_1+a_2}}A\right)y^{\frac{1}{a_1+a_2}-1}$$

$$=\left(p_1^{\frac{a_1}{a_1+a_2}}p_2^{\frac{a_2}{a_1+a_2}}A\right)$$

- So in the constant returns to scale case marginal cost doesn't depend on output
- This leaves us with three possibilities

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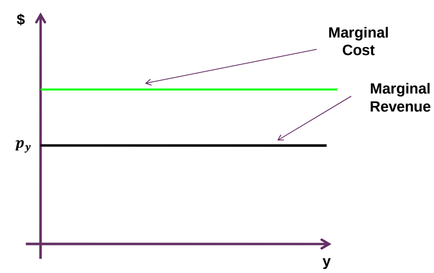
Marginal Revenue and Marginal Cost



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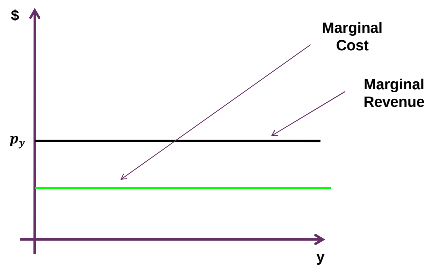
Marginal Revenue and Marginal Cost



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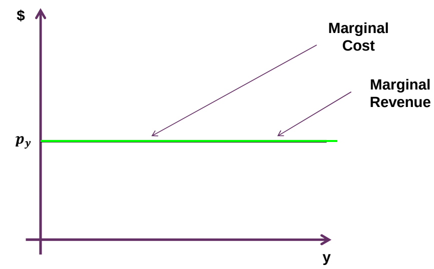
Marginal Revenue and Marginal Cost



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Profit Maximization

1. Marginal costs above price
   - Lose money on each item produced
   - Optimal to produce zero
2. Marginal costs below price
   - Make money on each item produces
   - Optimal to produce infinity
3. Marginal costs equal to price
   - Make 0 on each item produced
   - Profit equal to zero regardless of how much is produced

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Profit Maximization

- Putting this all together
- Diminishing returns to scale:
  - MC increasing
  - Optimal output is an **interior solution** as long as there is a y such that price equals marginal cost
  - What happens otherwise?
- Increasing returns to scale:
  - MC decreasing
  - Interior solution **minimizes** profit
  - Optimal output is a **corner solution –** to produce infinite output - as long as there is a y such that price is above marginal cost
  - What happens otherwise?
- Constant Returns to Scale
  - MC constant
  - Generally will not be an interior solution
  - Optimal output is a corner solution – either produce zero or infinity
  - Or anything if P=MC

Comparative Statics

What happens as prices change

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Comparative Statics

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- Remember, (one of) the reasons that we want to model firm behavior is to make **predictions**
- What type of predictions?
- Well, one thing we might want to do is figure out what happens when **input** and **output** prices change
  - Just as we did with the consumer
- E.g.
  - Do firms demand more labor if the price of their product goes up?
  - Do they produce more or less output if wages go up?
- This is what we mean by '**comparative statics'**

Comparative Statics

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- In order to do comparative statics we will make use of a new trick
- Lets say, that for prices p_y, p_1, p_2 the solution to the firm's problem is y, x_1, x_2
- And at prices $\overline{p_y}, \overline{p_1}, \overline{p_2}$ the solution to the firm's problem is $\bar{y}$, $\overline{x_1}, \overline{x_2}$
- What do we know?

Comparative Statics

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- It must be that, for prices p_y, p_1, p_2, inputs and outputs y, x_1, x_2 give higher profit than $\bar{y}$, $\overline{x_1}, \overline{x_2}$

$$p_y y - p_1 x_1 - p_2 x_2 \geq p_y \bar{y} - p_1 \overline{x_1} - p_2 \overline{x_2}$$

- Similarly

$$\overline{p_y}\bar{y} - \overline{p_1}\overline{x_1} - \overline{p_2}\overline{x_2} \geq \overline{p_y} y - \overline{p_1} x_1 - \overline{p_2} x_2$$

- Now, subtract the **right hand side** of the second equation from the **left hand side** of the first equation and the **left hand side** of the second equation from the **right hand side** of the first equation

$$\Delta p_y y - \Delta p_1 x_1 - \Delta p_2 x_2 \geq \Delta p_y \bar{y} - \Delta p_1 \overline{x_1} - \Delta p_2 \overline{x_2}$$

- Where $\Delta p_y = p_y - \overline{p_y}$, and so on

Comparative Statics

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$$\Delta p_y y - \Delta p_1 x_1 - \Delta p_2 x_2 \geq \Delta p_y \bar{y} - \Delta p_1 \overline{x_1} - \Delta p_2 \overline{x_2}$$

- Now subtract the **right hand side** of this equation from the left hand side

$$\Delta p_y \Delta y - \Delta p_1 \Delta x_1 - \Delta p_2 \Delta x_2 \geq 0$$

- This can allow us to make comparative static predictions

Comparative Statics

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- Prediction 1: what happens to output as the price of the output increases?

$$\Delta p_y \Delta y - \Delta p_1 \Delta x_1 - \Delta p_2 \Delta x_2 \geq 0$$

- Only prices of the output good change

$$\Delta p_y \Delta y \geq 0$$

- So if prices increase (and so $\Delta p_y > 0$) it must be the case that output (weakly) increases (i.e. $\Delta y \geq 0$)
- So output changes **in the same direction as** output prices

Comparative Statics

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- Prediction 2: what happens to input demand as the price of that input changes?
  - E.g. what happens to demand for labor (x_1) as the wage rate (p_1) changes?

$$\Delta p_y \Delta y - \Delta p_1 \Delta x_1 - \Delta p_2 \Delta x_2 \geq 0$$

- Only the 'wage rate' changes

$$-\Delta p_1 \Delta x_1 \geq 0$$

- So if wage rate increases (and so $\Delta p_1 > 0$) it must be the case that demand for labor (weakly) decreases (i.e.$\Delta x_1 \leq 0$)
- So input demand changes **in the opposite direction as** output prices
  - Caveat – this is a **simplified model**
  - **Do not** use this to go out and start voting against minimum wages!

Comparative Statics

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- What happens to the demand for labor as the price p_y changes?
- Previous approach of no use: x_1 could increase or decrease according to that equation
- What do you think?
- Well, we know that an increase in p_y leads to an increase in output
- Firm will move to a higher iso-output line – from y to $\bar{y}$
- Does this mean that they will use more of both inputs?

+ Fig 18: 'Inferior Inputs'

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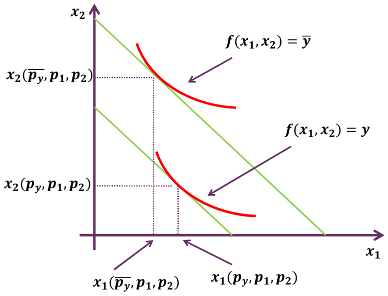

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+ Comparative Statics

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- An increase in p_y **will** increase output
- However, this may lead to a **fall** in the use of one of the inputs
- The logic is the same as in consumer theory for inferior inputs
- If you have to put up one shelf you might use a hand drill
- If you have to put up 10, you might go and buy an electric drill for $100
- Then putting up the 10 shelves might take you less time than putting up 1

+ Comparative Statics

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- What will be the impact of an **increase** in the price of one of the inputs on output
- For example, if labor becomes more expensive does this mean that the firm will produce less?
- Oddly enough, no
- Remember, what matters for output is **marginal cost**
- It is possible that an increase in p_1 will lead to an **increase** in total costs but a **fall** in marginal costs, leading to an increase in output

+ Fig 19: Marginal Revenue and Marginal Cost

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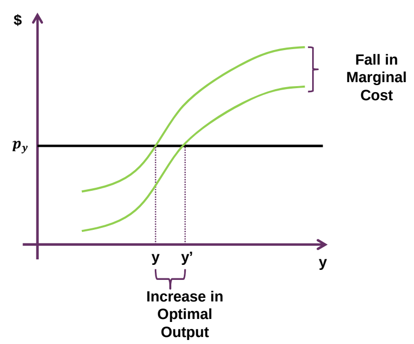

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+ Comparative Statics

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- How can this happen?
- First we need to know how to identify marginal costs in the cost minimization diagram

+ Marginal Cost in x_1, x_2 space

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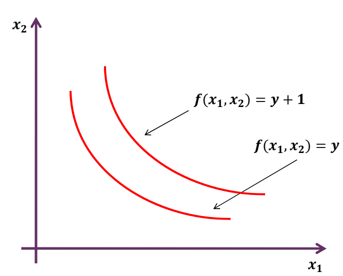

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Marginal Cost in x_1, x_2 space

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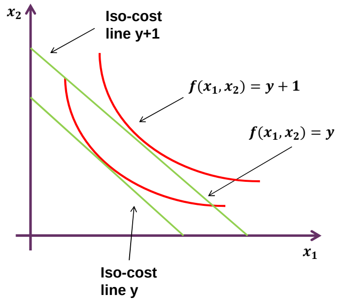

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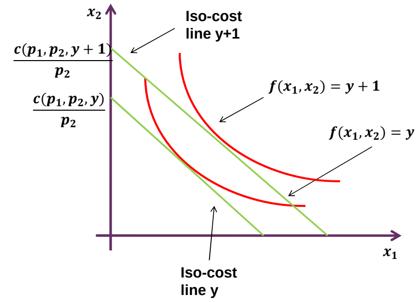
Marginal Cost in x_1, x_2 space

62



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Comparative Statics

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- How can this happen?
- First we need to know how to identify marginal costs in the cost minimization diagram
- We can read the 'marginal cost' of going from y to y+1 looking at where the minimum cost line intersects the vertical axis
  - Assuming p_2 does not change

An Increase in p_1 Leading to a Decreas in Marginal Cost

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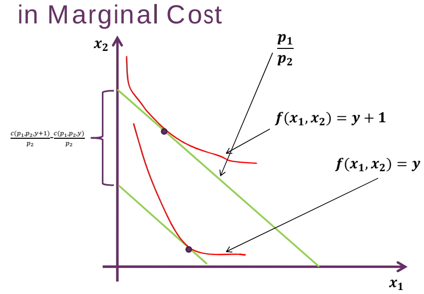

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An Increase in p_1 Leading to a Decreas in Marginal Cost

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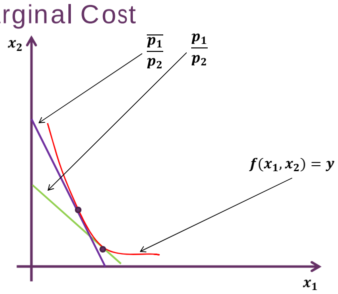

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An Increase in p_1 Leading to a Decreas in Marginal Cost

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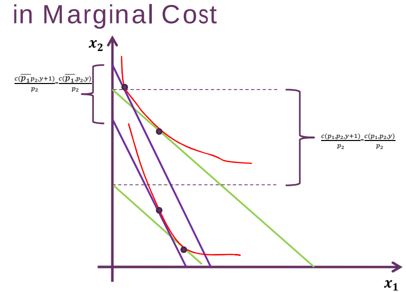

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Summary

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Summary

- Today we
- Described the relationship between returns to scale and cost functions
- Solved the second part of the firm's problem:
  - Choose the output level that maximizes profit given costs
  - i.e. the profit maximization problem
- Thought about comparative statics
  - i.e. what happens when we change the prices of inputs and outputs