Producer Theory - Perfect Competition

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1 Introduction

We have now given quite a lot of thought to how a consumer behaves when faced with different budget sets, and what happens in economies that consist only of consumers. However, as we have mentioned, there are several gaping holes in this analysis - one of which is that the real world does not consist only of consumers. We are now going to move to plug this gap by adding a new economic agent to our analysis - the firm. This is going to allow us to think about the supply of goods in a more interesting way than just having hermits who bring figs and brandy to an island.

For our purposes, a firm is going to be a machine that has the magical ability to convert on type of good (which we will call **inputs**) into another type of good (which we will call **outputs**). These inputs may be physical things, for example the raw materiels needed to produce something, but we will also allow for inputs to be more abstract things such as labor and capital. For example, a power station has the ability to change coal and the hard work of its employees into electricity, a university can transform the hard work of its professors, plus use of its classrooms into education and so on.

We are going to start off by thinking of a world in which firms (like our consumers before them) are *price takers*.There is a market price for the inputs they need and the outputs they want to sell, and the firm decides how much to buy and how much to produce. They can buy as many inputs, and sell as much output, as they like at these market prices. We describe such firms as **perfectly competitive.** In the next section we will relax this assumption to allow for the presence of a **monopoly**, who can choose what price to charge. After that, we will think about possibly the most interesting case, that of **oligopoly,** where we imagine that there are a small number of firms who will each be affected by the price that other firms charge. It is to figure out what is going on in this case that we will need to learn the tools of game theory.

2 The Optimization Problem of a Perfectly Competitive Firm

Remember how I spent an inordinate amount of time going on about optimization problems when we talked about the behavior of the consumer? We are now going to reap some of the benefits of doing so, because it turns out that we can also think of firms as solving their own optimization problem, and therefore we can use some of the same materiel that we used when modelling the consumer. Hurrah!

As I am sure you remember an optimization problem consists of the following elements:

> **Choose** <*some object*> **in order to maximize** <*some objective function*> **subject to** <*some constraint*>

So we need to decide what it is the firm gets to choose, what they are trying to maximize, and what are their constraints.

First of all, what is it that firms get to choose? As we discussed above, we are going to start off thinking about perfectly competitive firms, so the one thing they do *not* get to choose is price. What they do get to choose is their level of output (how much they get to produce) and their inputs (how much raw materiels they get to buy). For simplicity, we are going to think of a firm that produces only one good, but may have more than one input. Let's call x the amount of output, and k and l the amount of two different inputs (we can think of them standing for capital and labor). For simplicity, we will think of firms who at most use two inputs.

Now, what is it that the firm wants to maximize? Remember, it took us two lectures to talk through this issue for consumers. Luckily, this is much easier for firms: as any good capitalist knows, the aim of firms is to maximizer profits! Profits are described by the following expression

$$\pi = p_x x - wl - rk$$

where p_x is the price the firm can sell their good at, w is the price that the firm can buy l at and r is the price that the firm can buy k at (we can think of this as the wage rate for labor and the rental rate of capital). Thus, the profit that the firm makes (π) is equal to the revenue it makes from selling its product ($p_x x$) minus the amount that they have to spend on inputs to make that output.($wl + rk$).

What are the constraints that the firm operates under? Well, clearly there must be some link between the amount of the inputs that the firm buys, and the amount of output they can produce. We call this the **technology** of the firm, and it is summarized by the production function

$$x = f(l, k)$$

This function tells us the amount that the firm can produce if they use l amount of input 1 and k amount of input 2.

Thus we are now in a position to write down the optimization problem of the firm (in somewhat quicker time than it took us to write down the optimization function of the consumer):

Choose: *an output level x and levels of inputs l, k*

In order to maximize: *profits:* $\pi = p_x x - wl - rk$

Subject to: *the production function* $x = f(l, k)$

Here the **parameters** of the problem are the price of the good, p_x, and the price of the two inputs w and r. The inputs k and l are sometimes called the **factors of production**

3 The Case of One Input

We are going to begin by simplifying matters even further, by thinking of a firm that only requires one input of production - think of a firm that hires one person to dig holes, so the only input they have is the number of workers they are going to hire (let's assume that these workers come with their own shovels). The optimization problem therefore becomes

Choose: *Levels of input l*

In order to maximize: *profits:* $\pi = p_x x - wl$

Subject to: *the production function* $x = f(l)$

In order to make further progress with this problem, we are going to have to make some assumptions about the nature of the production function. In particular, we are going to assume three things

1. $f(l) \geq 0$
2. $\frac{\delta f(l)}{\partial l} > 0$
3. $\frac{\partial^2 f(l)}{\partial l^2} < 0$

Between them, these three assumptions guarantee that the production function looks like it does in figure 1.

The first assumption says that, for any amount of labour input, the output is going to be 0 or bigger - in other words we cannot have negative production. This seems fairly natural

The second assumption tells us that the first derivative of the production function has to be positive. We call the first derivative of the production function the **marginal product** (with respect to the factor of production) - it measures how production will change if we add an additional unit of labor. The second assumption says that if we buy more labor then we will get more output. This seems relatively uncontroversial, and means that the production function is upward sloping.

The third assumption is slightly more controversial: it says that the rate at which adding labor increases output falls as we get more labor: in other words, the additional output we get in going from 1 to 2 units of labor is higher than the additional output we get going from 2 to 3 units of labor and so on. In other words, the production function is *concave.* Another way to say the same thing is that there is **diminishing marginal productivity of labor.** There are a few different ways to think about this. One is that labor actually varies in quality - so the first worker that the firm hires is better than the second one and so on. Thus, productivity falls as the firm hires more (and worse) workers. A second is that the firm only has a fixed number of shovels (and for some reason cannot buy any more) - so while the first worker gets to use the shovel all the time, when there are two workers they have to share the shovel and so on. However, one can clearly think of cases where diminishing marginal productivity makes little sense. For example, when it comes to moving furniture, it may be that two people are more than twice as productive as one person. Whether marginal productivity rises or falls is clearly an empirical question, but at the moment we are going to maintain the assumption of diminishing marginal productivity for convenience. .

How do we solve the firm's optimization problem? Actually, we are going to think of three different ways of doing so. The first way is akin to the way that we set about solving the consumer's problem. There, we wanted to get onto the highest possible indifference curve, subject to being in the budget set. Here we want to be on the highest profit iso-profit line consistent with being on the production function.

What do iso-profit lines look like? Well, we know that

$$\pi = p_x x - wl$$

so rearranging gives us

$$x = \frac{\pi}{p_x} + \frac{w}{p_x} l$$

Thus, iso-profit lines are upward sloping, with a slope equal to $\frac{w}{p_x}$. Also, moving in a North-Easterly direction moves us to higher iso-profit lines (as the intercept with the vertical axis is given by $\frac{\pi}{p_x}$). Figure 2 shows what iso profit lines look like, and figure 3 shows what the profit maximizing output is going to look like. As we might suspect (assuming that we have an interior solution) profit maximization is going to occur at the point of tangency - n this case between the production function and the iso-profit line. And what is true at the point of tangency? The slope of the budget line has to equal the slope of the production function. In other words, if l^* is the profit maximizing level of labor input, then

$$\frac{\partial f(l^*)}{\partial l} = \frac{w}{p_x}$$

or, put another way, the marginal productivity has to equal the ratio of the wage rate to the price of the good x.

Are our assumptions enough to guarantee an interior solution? Unfortunately not. It could be the case that the iso profit lines are always steeper than the production function, or that they are always shallower. In the former case, the firm will want to produce 0 goods, while in the latter they will want to produce an infinite amount. You will deal with such cases in the homework.

A second way to get to the same conclusion is to just solve the optimization problem directly. In order to do so, realize that we can turn the constrained optimization problem of the firm into an unconstrained problem by getting rid of the variable x altogether. As soon as we have chosen

l, we have automatically chosen x, as the production function tells us what x has to be. Thus, we can rewrite the problem as

Choose: *Levels of input* l, k

In order to maximize: *profits:* $\pi = p_x f(l) - wl$

And we know how to solve this using calculus Assuming the second order conditions are satisfied, then this just means differentiating the profit function and setting the result equal to zero:

$$\begin{aligned} \frac{\partial \pi}{\partial l} &= \frac{\partial \left(p_x f(l) - wl\right)}{\partial l} \\ &= p_x \frac{\partial f(l)}{\partial l} - w = 0 \\ &\Rightarrow \frac{\partial f(l)}{\partial l} = \frac{w}{p_x} \end{aligned}$$

This is, unsurprisingly, the same condition as we have before.

Note that, given our assumptions, the second order conditions will be satisfied, as they state that, for this to be a maximum, the second derivative has to be negative: in other words

$$\frac{\partial^2 \pi}{\partial l^2} = p_x \frac{\partial^2 f(l)}{\partial l^2} < 0$$

Luckily, if we go back and have a look at the assumptions we made about the production function, we have already guaranteed that this is the case!

Another way of getting intuition into this result is to realize that we can split the profit function up into cost and revenue: revenue equals the amount that is produced times the price charged per unit: i.e. $p_x f(l)$, while the cost equals the price per unit of input times the amount of input, i.e. wl. Thus

$$\pi = \text{revenue-cost}$$

Thus, the first order condition that the derivative of the profit function equals zero is the same as saying that marginal benefit of hiring an extra unit of labor is equal to the marginal cost of doing so. The former is just the extra production that the additional unit of labor gives times the price that the unit can be sold at, i.e. $p_x \frac{\partial f(l)}{\partial l}$, while the latter is just equal to w. Thus, we can see the profit maximizing condition in the form of the graph in figure 4.

We can use the graph in figure 4 to also see what will happed to optimal output and labor as wages and prices change. Figure 5 shows the effect of an increase in the wage rate. This has the effect of increasing the marginal cost of employing another worker. Remember that, at the optimum, this has to equal the marginal revenue of employing another worker. Because of our assumptions about the production function, the firm increases its marginal revenue per worker by reducing the number of workers, and therefore reducing labor used and output.

We can see this mathematically in the following way. Let $l^*(w, p_x)$ be the optimal level of labor given prices p and wages w. We know that

$$\frac{\partial f(l^*(w, p_x))}{\partial l} = \frac{w}{p_x}$$

so, differentiating by w tells us that

$$\frac{\partial^2 f(l^*(w, p_x))}{\partial l^2} \frac{\partial l^*(w, p_x)}{\partial w} = \frac{1}{p_x}$$

As $\frac{1}{p_x}$ is positive and $\frac{\partial^2 f(l^*(w,p_x))}{\partial l^2}$ is negative (by assumption), it must be the case that $\frac{\partial l^*(w,p_x)}{\partial w}$ is also negative.

Figure 6 shows what happens in the price of the good x increases. This raises the marginal revenue of hiring an extra worker at any given level of production. Thus, in order to make the marginal revenue of employing another person equal the marginal cost of doing so, the firm has to increase production, reducing marginal productivity. Again, we can see this mathematically. Differentiating $\frac{\partial f(l^*(w,p_x))}{\partial l} = \frac{w}{p_x}$ with respect to p_x gives

$$\frac{\partial^2 f(l^*(w, p_x))}{\partial l^2} \frac{\partial l^*(w, p_x)}{\partial p_x} = -\frac{w}{p_x^2}$$

So, as $-\frac{w}{p_x^2}$ is negative, and $\frac{\partial^2 f(l^*(w,p_x))}{\partial l^2}$ is negative, $\frac{\partial l^*(w,p_x)}{\partial p_x}$ must be positive.

In order to fix ideas, let us do a specific example: lets assume that $f(l) = l^{\frac{1}{2}}$ (you should check that this satisfies our three assumptions). The profit for employing l workers is therefore

$$\begin{aligned} \pi &= p_x f(l) - wl \\ &= p_x l^{\frac{1}{2}} - wl \end{aligned}$$

Taking derivatives with respect to l gives

$$\begin{aligned}\frac{\partial \pi}{\partial l} &= \frac{1}{2} p_x l^{-\frac{1}{2}} - w = 0 \\ &\Rightarrow \ l = \left(\frac{p_x}{2w}\right)^2\end{aligned}$$

This is the optimal amount of labor demand as a function of wages and prices. Plugging this back into the production function gives us

$$x = f(l) = l^{\frac{1}{2}} = \frac{p_x}{2w}$$

This is the **supply function,** or how much a firm will choose to supply at for any level of prices and wages. As we would expect, it is upward sloping, so firms will choose to supply more of the good as prices increase.

We are now going to look at a third way of solving the same problem. Annoying as this may seem, the insights we get here will come in very handy further down the road. We are going to solve the firm's optimization problem by getting rid of l, or labor, rather than getting rid of x the amount of the good produced. Remember that

$$x = f(l)$$

So it is also the case that

$$l = f^{-1}(x)$$

where f^{-1} is just the inverse of the demand function (if you are not sure how inverses work, then the website http://uncw.edu/courses/mat111hb/functions/inverse/inverse.html has a good description).

Therefore we can rewrite the firm's profit function as

$$\pi(x) = p_x x - w f^{-1}(x)$$

Now the first term is the revenue gained from selling x units, while the second term is the cost of producing those x units. In fact, we often rewrite $wf^{-1}(x)$ as $c(w, x)$ - the cost of producing x units when the wage rate is w, so the profit function becomes

$$\pi(x) = p_x x - c(w, x)$$

The optimal level of output is found (assuming that the second order conditions are satisfied) by setting the derivative with respect to x to zero:

$$\begin{aligned} p_x - \frac{\partial c(w,x)}{\partial x} &= 0 \\ &\Rightarrow \; p_x = \frac{\partial c(w,x)}{\partial x} \end{aligned}$$

In other words, the **marginal revenue** (which we will write as $MR(x)$) equals the **marginal cost (**$MC(x)$) of producing another unit of production.

Note that the terminology gets a bit confusing here, as we have used terms that sound similar to marginal revenue and marginal cost before. However, then we were talking about marginal revenue and marginal cost *with respect to changes in labor demand* in other words, the additional revenue and cost of employing one more unit of labor. Here, we are talking about marginal revenue and marginal cost *with respect to an extra unit of production* - in other words the extra revenue and cost of producing one more item to sell. If we are being formal, we should always say marginal revenue and marginal cost *with respect to what.* However, if it is not specified the people usually mean with respect to an extra unit of production rather than labor.

It is very important to note that there is a relationship between marginal productivity and marginal cost. To see this, remember that

$$c(w,x) = wf^{-1}(x)$$

So marginal cost is given by

$$\frac{\partial c(w,x)}{\partial x} = w\frac{\partial f^{-1}(x)}{\partial x}$$

A handy theorem called the inverse function theorem (that you may or may not know) is that, in general

$$\frac{\partial f^{-1}(x)}{\partial x} = \frac{1}{\frac{\partial f(l)}{\partial l}}$$

where $l = f^{-1}(x)$,so

$$\frac{\partial c(w,x)}{\partial x} = w\frac{\partial f^{-1}(x)}{\partial x} = \frac{w}{\frac{\partial f(l)}{\partial l}}$$

But $\frac{\partial f(l)}{\partial l}$ is just the marginal product, so there is an inverse relationship between marginal product and marginal cost: If marginal product is falling, then marginal cost is rising, and visa versa. This should also make perfect sense - if the amount we can produce with an additional unit of labor is falling, then the cost of producing an additional unit of output should be rising.

Our baseline assumption is that $\frac{\partial f(l)}{\partial l}$ is decreasing as x and l increases, which implies that $\frac{\partial c(w,x)}{\partial x}$ increases as x increases. The graph of marginal cost and marginal revenue will therefore look like it does in figure 7.

Going back to our example, in which $f(l) = l^{\frac{1}{2}}$ we can see that

$$\begin{aligned} x &= l^{\frac{1}{2}} \\ \text{and so } x^2 &= f^{-1}(x) = l \end{aligned}$$

The cost function is therefore given by

$$c(w,x) = wf^{-1}(x) = wx^2$$

and the production function

$$\pi = p_x x - wx^2$$

The first order condition gives us

$$\begin{aligned} \frac{\partial \pi}{\partial x} &= p_x - 2wx = 0 \\ &\Rightarrow x = \frac{p_x}{2w} \end{aligned}$$

Exactly the same result as when we replaced x rather than l..

Another important concept to introduce here is that of the **average cost**. This is just the average cost of each unit produced, or

$$AC(x,w) = \frac{C(x,w)}{x}$$

A couple of interesting things about average costs: First, they are related to marginal costs, in the following way: If marginal costs are higher than average costs must be rising, and visa versa. This makes sense: If the cost of an additional unit of production is higher that the average cost of things produced so far, then it *must* be the case that average cost is rising. We can see this formally by noting that

$$AC(x,w) = \frac{C(x,w)}{x}$$

and so

$$\begin{aligned}
\frac{\partial AC(x,w)}{\partial x} &= \frac{\frac{\partial C(x,w)}{\partial x}}{x} - \frac{C(x,w)}{x^2} \\
&= \frac{1}{x}\left[\frac{\partial C(x,w)}{\partial x} - \frac{C(x,w)}{x}\right] \\
&= \frac{1}{x}\left[MC(x) - AC(x,w)\right]
\end{aligned}$$

Which will be positive if and only if marginal cost is greater than average cost.

The second thing is that profit will be positive if and only if price is greater than average cost. To see this, note that

$$\begin{aligned}
\pi(x) &= p_x x - c(w,x) > 0 \\
&\Rightarrow p_x x > c(w,x) \\
&\Rightarrow p_x > \frac{c(w,x)}{x} = AC(x,w)
\end{aligned}$$

To illustrate some of the above points a little bit more, let's think of a production function that breaks one of our initial three assumptions- in particular a production function that is not always concave. In particular, let's think about the production function illustrated in figure 8. This shows a production function that is initially convex, and then becomes concave at higher levels of labor. - in figure 8 the production function is convex below L^* and concave above L^*. (One reason to study such a function is that they are very popular in more basic economic courses).

What does marginal product look like with such a production function? Well, remember that marginal product is just the slope of the production function, which is always positive. However, when the production function is convex, marginal product is increasing (i.e. for labor below L^*), while for values above L^* it is decreasing. In other words, for values below L^*, labor is becoming more productive the more is hired, while for values above L^* it is becoming less productive. This is shown in figure 9.

Before, we said that profit is maximized when the iso profit lines are tangent to the production function. However, in this case we are going to get two points of tangency. as shown in figure 10, labelled L_1 and L_2. Obviously only one of these is profit maximizing. In fact, L_1 is a profit minimizing output.

What does marginal cost look like in this case? Well, for output below x^* (the output given by l^*), marginal product is increasing so, as we have shown above, marginal cost is falling. For output above x^*, marginal product is falling so marginal cost is increasing. Thus, marginal and average cost look as they do in figure 11. Note that marginal cost crosses average cost at the bottom point of its curve (why?).

One very important point is that we need marginal product to start falling at some point if we are going to be able to solve the firms problem. To see this, note that, if marginal product is always rising, then marginal cost is always falling - thus a firm who is profitable at a certain level of output will be even *more* profitable if they produce more. You will work more with this problem in the homework.

4 The Case of Two Inputs

Now we have dealt with the case where there is only one input into the firm, we will now go back to the more general (and more interesting) case of two inputs. In other words, the firm's optimization problem is once again

> **Choose:** *an output level x and levels of inputs l, k*
>
> **In order to maximize:** *profits:* $\pi = p_x x - wl - rk$
>
> **Subject to:** *the production function* $x = f(l, k)$

Once again, we have two potential methods to solve this problem.

Method 1 Substitute in for x and make it an unconstrained optimization problem in l and k. In other words, choose l and k to maximize

$$\begin{aligned} \pi &= p_x x - wl - rk \\ &= p_x f(l, k) - wl - rk \end{aligned}$$

Method 2 Split the problem up into two parts:.

1. Find the cheapest way to produce any amount x, and so calculate $c(x, r, w)$, or the cost of producing x when rent is r and wages are w
2. Find the level of output that maximizes

$$\begin{aligned} \pi &= p_x x - wl - rk \\ &= p_x x - c(x, r, w) \end{aligned}$$

It turns out that the second approach is going to be easier, provide more insight, and allows us to use many of the tools that we have already developed, so we will concentrate on that. The other way, is of course not impossible, and you can try it if you wish.

Taking the second approach, we now have two optimization problems to solve, the first being a cost minimization problem:

Choose: *a levels of inputs* l, k

In order to minimize: *costs* $wl + rk$

Subject to: attaining some level of output $\bar{x} = f(l, k)$

We can think of solving this problem graphically, using a graph that has k on the horizontal axis and l on the vertical axis. We can then draw iso-cost lines to represent our objective functions, and an iso-output lines to represent the constraint. As we do so, things should start to look hauntingly familiar.

First, what do iso cost lines look like? We can get that just from rearranging the following expression

$$\begin{aligned} c &= wl + rk \\ &\Rightarrow l = \frac{c}{w} - \frac{r}{w}k \end{aligned}$$

So iso cost lines are straight lines, with slope $\frac{r}{w}$, and intercept the vertical axis at $\frac{c}{w}$, as shown in figure 12. Moreover, we get to lower cost lines as we move in a South-Westerly direction.

What does the constraint look like? This is going to depend on the properties of the production function. The constraint says that we have to use enough k and l in order to produce an amount $\bar{x}$. We therefore want to draw an iso-output line on our graph, or the collection of k and l that produce an output $\bar{x}$. We are going to make two assumptions about the production function that should again remind you of something

1. It is monotonic: i.e. for $l_2 \geq l_1$ and $k_2 \geq k_1$ $f(l_2, k_2) \geq f(l_1, k_1)$

2. It is convex i.e.

$$\begin{aligned} \text{For } l_1, k_1, l_2, k_2 \text{ such that } f(l_1, k_1) &= f(l_2, k_2) \\ &\text{and number } \lambda \text{ between 0 and 1} \\ f((\lambda l_1 + (1-\lambda) l_2), (\lambda k_1 + (1-\lambda) k_2) &\geq f(l_1, k_1) \end{aligned}$$

As usual we can define strict monotonicity and strict convexity in the appropriate way.

What do these assumptions mean? Well, the first one is quite intuitive - it just tells us that if we employ more k and l then we get (at least weakly) more output. This seems pretty uncontroversial.

As usual, the second assumption is a little bit more controversial, but also a little bit more useful. What this assumption is telling us (roughly) is that we are going to get more production if we employ 5 units of l and 5 units of k than if we employ 10 units of l and 0 of k, or 0 units of l and 10 of k. In other words, it is more efficient to use 'average bundles' of inputs than it is to use extreme bundles.

What does this imply for our graph of the production function? You should be able to tell if you realize that the assumptions we just made about the production function are exactly the same as those that we made about the utility function. Moreover, an iso-output line is exactly the same as an indifference curve. Thus, our iso output lines are going to look exactly the same as our indifference curves, in that they are

1. Downward sloping
2. Do not cross
3. Move to higher levels of production as we move in a North Easterly direction
4. Are convex

In fact, we tend to use the same types of function to represent production functions as we do to represent utility functions namely

1. Cobb-Douglas production functions:

$$f(k,l) = k^{\alpha} l^{\beta}$$

2. Perfect compliment production functions (usually called Liontief production functions)

$$f(k,l) = \min(ak, bl)$$

3. Perfect substitute production functions

$$f(k,l) = ak + bl$$

These are illustrated in figures 13, 14 and 15.

So the problem is one of figuring out the lowest possible iso cost curve we can get onto, given that we have to be on a particular iso output line, as illustrated in figure 16. Luckily, this is exactly the same problem that we have spent ages solving in consumer theory, so we know exactly what we are trying to do here (yes?). In fact, exactly the same procedure will work here as it did in finding the optimal bundle for the consumer.

1. Find any points of tangency
2. Calculate the cost of production at the points of tangency
3. Compare to the cost of production at the corner solutions (if corner solutions are possible)
4. Pick the point of tangency or corner solution that has the lowest cost of production

By the same logic we used for the consumer, we know that any solution to this problem has to be either a point of tangency or a corner solution, so solving this problem is just the case of choosing the best from this set. The only wrinkle here is that it may, in fact not be possible to obtain a particular level of output at a corner solution (for example, is it possible to produce 10 units of output using only k if the production function is Liontief?)

How do we find points of tangency? Again, the process should be familiar. The point of tangency is the point at which the slope of the iso-cost line equals that of the iso output line. The slope of the former is easy - it is just $-\frac{r}{w}$, or the ratio of the rental rate to the wage rate. What about the slope of the iso-output line? Remember that all the points along an iso out put line are combinations of labor and capital that give the same level of output. Thus, the slope of the iso-output line measures the rate at which you can substitute labor for capital while keeping the level of output constant (as shown in figure 17). We call this the **marginal rate of technical substitution**. Note the parallels between this concept and the marginal rate of substitution that we discussed in consumer theory. There, we were measuring how we could trade off one good for another, while keeping the consumer indifferent. Here, we are measuring how we can trade off one good for another while keeping the level of output constant.

Thus we have our cost minimizing condition (assuming we have an interior solution)

$$\frac{r}{w} = MRTS_{l,k}$$

Where $MRTS_{l,k}$ is the marginal rate of technical substitution between labor and capital.

How do we find the marginal rate of technical substitution? A quick think back to the case of the consumer might lead us to think that it is something to do with the derivative of the production function with respect to capital and labor. And indeed this is the case. We can see this by totally differentiating the production function

$$df(l,k) = \frac{\partial f(l,k)}{\partial k}dk + \frac{\partial f(l,k)}{\partial l}dl$$

As, along the iso output curve we know that $df(l,k) = 0$, we have

$$\begin{aligned} \frac{\partial f(l,k)}{\partial k}dk + \frac{\partial f(l,k)}{\partial l}dl &= 0 \\ \Rightarrow \frac{dl}{dk} &= -\frac{\frac{\partial f(l,k)}{\partial k}}{\frac{\partial f(l,k)}{\partial l}} = MRTS_{l,k} \end{aligned}$$

Thus, the marginal rate of technical substitution is equal to the ratio of the marginal products.

We now know how to find the cost minimizing use of capital and labor with respect to the parameters of the problem. We will now illustrate this with an example:

Example 1 *We want to find the profit maximizing output of a firm that can sell its goods for price* p_x*, faces factor prices* w *and* r *and has a production function of the form* $f(k,l) = k^{\alpha}l^{\beta}$.

The first thing we need to do is find the cost function $c(w,r,x)$ *of the firm, or in other words the cost minimizing way of producing* x *goods. As this is a Cobb-Douglas production function we know that this is going to be an interior solution, and therefore cost minimization will occur at the point where* $MRTS_{l,k} = \frac{r}{w}$*. We also know that* $\frac{\frac{\partial f(l,k)}{\partial k}}{\frac{\partial f(l,k)}{\partial l}} = MRTS_{l,k}$*, so we will begin by taking the derivative of the production function with respect to labor and capital*

$$\begin{aligned} \frac{\partial f(l,k)}{\partial k} &= \alpha k^{\alpha-1}l^{\beta} \\ \frac{\partial f(l,k)}{\partial l} &= \beta k^{\alpha}l^{\beta-1} \end{aligned}$$

Taking the ratio of the two marginal products gives us

$$\frac{\frac{\partial f(l,k)}{\partial k}}{\frac{\partial f(l,k)}{\partial l}} = \frac{\alpha k^{\alpha-1}l^{\beta}}{\beta k^{\alpha}l^{\beta-1}} = \frac{\alpha}{\beta}\frac{l}{k}$$

and setting this equal to the ratio of rent to wages gives

$$\begin{aligned}\frac{\alpha}{\beta}\frac{l}{k} &= \frac{r}{w} \\ &\Rightarrow\ k = \frac{\alpha}{\beta}\frac{w}{r}l\end{aligned}$$

So this now tells us the ratio of k *to* l *at the cost minimizing point. But how much* k *and* l *do we use to produce output* x*? To figure this out we need to substitute into the production function*

$$\begin{aligned}f(k,l) &= x \\ &\Rightarrow\ k^{\alpha}l^{\beta} = x \\ &\Rightarrow\ \left(\frac{\alpha}{\beta}l\frac{w}{r}\right)^{a} l^{\beta} = x \\ l^{\alpha+\beta} &= \left(\frac{r}{w}\frac{\beta}{\alpha}\right)^{\alpha} x \\ l &= \left(\frac{r}{w}\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}} x^{\frac{1}{\alpha+\beta}}\end{aligned}$$

Substituting back in, we get that demand for capital is given by

$$\begin{aligned}k &= \frac{\alpha}{\beta}\frac{w}{r}l \\ &= \left(\frac{r}{w}\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}-1} x^{\frac{1}{\alpha+\beta}} \\ &= \left(\frac{\alpha}{\beta}\frac{w}{r}\right)^{\frac{\beta}{\alpha+\beta}} x^{\frac{1}{\alpha+\beta}}\end{aligned}$$

We therefore now know the demand for labor and capital as a function of r*,* w *and* x*. We will write these demand functions as*

$$\begin{aligned}k^{*}(r,w,x) &= \left(\frac{\alpha}{\beta}\frac{w}{r}\right)^{\frac{\beta}{\alpha+\beta}} x^{\frac{1}{\alpha+\beta}} \\ l^{*}(r,w,x) &= \left(\frac{r}{w}\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}} x^{\frac{1}{\alpha+\beta}}\end{aligned}$$

But remember what we are really interested in is the cost function. However, this is easy to calculate, as we know that costs are just given by

$$\begin{aligned}c(r,w,x) &= wl^{*}(r,w,x) + rk^{*}(r,w,x) \\ &= w\left(\frac{r}{w}\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}} x^{\frac{1}{\alpha+\beta}} + r\left(\frac{\alpha}{\beta}\frac{w}{r}\right)^{\frac{\beta}{\alpha+\beta}} x^{\frac{1}{\alpha+\beta}} \\ &= \left(w^{\frac{\beta}{\alpha+\beta}} r^{\frac{\alpha}{\alpha+\beta}} \left(\left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}} + \left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}}\right)\right) x^{\frac{1}{\alpha+\beta}}\end{aligned}$$

This gives us the cost function for our producer.

So we now have solved half the problem for our firm - for any given level of output we can figure out the cheapest way of producing that level of output. The next thing we need to do is to figure out the profit maximizing level of output. Recall, this will occur at the point at which price equals marginal cost, as

$$\pi(x) = p_x x - c(w, r, x)$$

and so, for profit maximization we have

$$p_x = \frac{\partial c(w, r, x)}{\partial x}$$

IF WE HAVE AN INTERIOR SOLUTION.

Recall that, in the case of a single factor of production, we said that to find an interior solution it had to be the case that marginal costs have to start rising at some point: If marginal costs are always falling, then the firm will either want to produce 0 or ∞. This is also the case with multiple inputs: if the firm is not going to be at a corner solution, we need marginal costs to start rising at some point.

In the case of a single input, all we needed for marginal costs to start rising was for marginal product to start falling at some point. Is that enough in the case of multiple inputs? To answer this, lets have a look at our example. Remember that marginal product of capital is given by

$$\frac{\partial f(l, k)}{\partial k} = \alpha k^{\alpha - 1} l^{\beta}$$

And so to find out whether the marginal product of capital is increasing or decreasing, we take the second derivative of the production function:

$$\frac{\partial^2 f(l, k)}{\partial k^2} = \alpha(\alpha - 1) k^{\alpha - 2} l^{\beta}$$

This will be decreasing as long as $\alpha < 1$. Similarly, the marginal product of labor will be decreasing if $\beta < 1$. So, if both α and β are less than one, is that enough to guarantee that that marginal cost is rising? Remember that the cost function is given by

$$c(r, w, x) = \left(w^{\frac{\beta}{\alpha+\beta}} r^{\frac{\alpha}{\alpha+\beta}} \left(\left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}} + \left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} \right) \right) x^{\frac{1}{\alpha+\beta}}$$

For convenience, let's call $\left(w^{\frac{\beta}{\alpha+\beta}}r^{\frac{\alpha}{\alpha+\beta}}\left(\left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}+\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}}\right)\right)=A$, and note that A has to be positive. So marginal cost is given by

$$\frac{\partial c(r,w,x)}{\partial x}=\frac{1}{\alpha+\beta}Ax^{\frac{1}{\alpha+\beta}-1}$$

To figure out whether marginal costs are increasing or decreasing, we differentiate again with respect to x, giving

$$\frac{\partial^2 c(r,w,x)}{\partial x^2}=\left(\frac{1}{\alpha+\beta}\right)\left(\frac{1}{\alpha+\beta}-1\right)Ax^{\frac{1}{\alpha+\beta}-2}$$

Marginal costs will be increasing as long as

$$\begin{aligned}\frac{1}{\alpha+\beta}-1 &> 0\\ &\Rightarrow 1>\alpha+\beta\end{aligned}$$

So we require the *sum* of α and β to be less that 1. Thus, it is not enough for marginal product to be decreasing: if $\alpha=0.75$ and $\beta=0.75$ then the marginal product of capital and labor will be decreasing, but marginal costs will always fall.

So if marginal product is not enough to tell us whether marginal costs are rising or falling, what is? It turns out, we need to introduce the idea of the **returns to scale** of a firm. The idea of returns to scale is to ask the following question. If we doubled the amount of each input into production, what would happen to output? We can basically think of three cases: Firstly, it could be the case that doubling the inputs *more* than doubles the output: This is the case in which a firm becomes more efficient as it increases in size. Second, it could be the case that doubling the inputs *less* than doubles output. This would be the case if the firm were to get *less* efficient as it got larger. Alternatively, it could be the case that doubling inputs *exactly* doubles outputs. In this case, the size of the firm does not affect its efficiency.

The first of these cases is an example of increasing returns to scale, the second decreasing returns to scale, the third constant returns to scale. In fact, we define the concepts slightly more generally than just doubling the inputs, as we can see below:

Definition 1 *Let $f(l,k)$ be the production function of a firm if, for $t>1$*

1. *We say that the firm exhibits increasing returns to scale if $tf(l,k)<f(tl,tk)$*

2. *We say that the firm exhibits decreasing returns to scale if* $tf(l,k) > f(tl,tk)$

3. *We say that the firm exhibits constant returns to scale if* $tf(l,k) = f(tl,tk)$

What is the returns to scale of the Cobb-Douglas firm that we used in our example? We can work this out as follows:

$$\begin{aligned} f(tk,tl) &= (tk)^{\alpha}(tl)^{\beta} \\ &= t^{\alpha+\beta}k^{\alpha}l^{\beta} \\ &= t^{\alpha+\beta}f(k,l) \end{aligned}$$

In other words, the firm will exhibit increasing returns to scale if $\alpha+\beta > 1$, decreasing returns to scale if $\alpha+\beta < 1$ and constant returns to scale if $\alpha+\beta = 1$. But if we look back at the conditions for whether or not marginal costs are increasing and decreasing, we see that marginal costs are decreasing only if $\alpha+\beta < 1$. So in other words, if the firm exhibits decreasing returns to scale, then their marginal costs will be rising. This will generally be true for the problems we look at: decreasing marginal product is not enough to guarantee rising marginal costs, but decreasing returns to scale is.

Now lets carry on with our example, assuming that $\alpha+\beta < 1$

Example 2 (Example 1 continued) *Now that we have solved the cost minimization problem, we can solve for the profit maximizing output. Remember that*

$$\pi = p_x x - c(r,w,x)$$

so the profit maximizing condition is that

$$\frac{\partial \pi}{\partial x} = p_x - \frac{\partial c(r,w,x)}{\partial x} = 0$$

Looking back at the cost function, we see that

$$\frac{\partial c(r,w,x)}{\partial x} = \frac{1}{\alpha+\beta}Ax^{\frac{1}{\alpha+\beta}-1}$$

And so

$$\begin{aligned} p_x &= \frac{1}{\alpha+\beta}Ax^{\frac{1}{\alpha+\beta}-1} \\ &\Rightarrow \quad x = \left(\frac{p_x}{A}(\alpha+\beta)\right)^{\frac{\alpha+\beta}{1-\alpha-\beta}} \end{aligned}$$

where, by the assumption of decreasing returns to scale, $\frac{\alpha+\beta}{1-\alpha-\beta} > 0$. Having solved for x, we can now plug back into the demand functions for labor and capital to solve for the input demand.

A final issue is the *comparative statics* of the firm: In other words - how do changes in the parameters of the problem (p_x, w and r) affect the amount of output that the firm chooses to produce, and its demand for inputs? To answer this question, we are going to pull a new trick: Lets imagine that $p_{x,1}$, w_1 and r_1 and one set of parameters, and x_1^*, l_1^* and k_1^* are the optimal output and input demands associated with those parameters. Similarly, let $p_{x,2}$, w_2 and r_2 be a second set of parameters, and x_2^*, l_2^* and k_2^* be the optimal response to those parameters. If x_1^*, l_1^* and k_1^* are profit maximizing for $p_{x,1}$, w_1 and r_1, then they must give more profit than x_2^*, l_2^* and k_2^* for those parameters. In other words

$$p_{x,1}x_1^* - w_1 l_1^* - r_1 k_1^* \geq p_{x,1}x_2^* - w_1 l_2^* - r_1 k_2^*$$

Similarly,

$$p_{x,2}x_2^* - w_2 l_2^* - r_2 k_2^* \geq p_{x,2}x_1^* - w_2 l_1^* - r_2 k_1^*$$

Subtracting the right hand side of the second equation from the left hand side of the first equation, and the left hand side of the second equation from the right hand side of the top equation gives

$$\Delta p_x x_1^* - \Delta w l_1^* - \Delta r k_1^* \geq \Delta p_x x_2^* - \Delta w l_2^* - \Delta r k_2^*$$

Where $\Delta w = w_1 - w_2$ and so on. This tells us that

$$\Delta p_x \Delta x^* - \Delta w \Delta l^* - \Delta r \Delta k^* \geq 0$$

This equation immediately tells us something about the comparative statics of the problem. First, assume that the price of good x increases, but wages and rents do not change. The above equation tells us that $\Delta p_x \Delta x^* \geq 0$, so if $\Delta p_x > 0$, it must be the case that $\Delta x^* \geq 0$: In other words, if the price increases then the amount the firm produces will have to increase. Similarly, if the wage rate goes up, but prices and rents do not change, then the above equation tells us that $\Delta w \Delta l^* \leq 0$, so if wages go up, then demand for labor must go down.

So far, so sensible. But what is the effect of an increase in prices on factor demand? Or an increase in the wage rate on output? The results here are slightly less intuitive. Firstly, it can be the case that an *increase* in the price p_x can lead to a *decrease* in demand for one of the inputs of production. To see this, look at figure 18. We know from above that an increase in p_x will lead to an increase in output, say from x to x^*. Thus, the firm moves to a higher iso-output line, as indicated in the figure. However, it turns out that the cost minimizing way of producing x^* actually uses *less* capital than the cost minimizing way of producing x. Thus, and increase in p_x reduces demand for k. In fact, this should not be *that* surprising - this is just the same effect that gave us inferior goods in consumer theory.

What is perhaps less intuitive is that an increase in the price of one of the inputs can actually lead to an *increase* in production. This is because, while such an increase must lead to an increase in total costs, it can lead to a *fall* in marginal costs. Figure 19 shows that a change in factor costs that leads everywhere to a fall in marginal costs will lead to an increase in optimal output.

So how can an increase in a factor price lead to a fall in marginal cost? In order to answer this question, we need to learn how to read marginal costs of a graph of the type we introduced in figure 12. To see this, look at figure 20. This figure shows two iso-output lines, one for an amount x, and one for an amount $x+1$. It also shows the minimum cost lines for each level of output, which are labelled iso-cost line x and iso cost line $x+1$. Remember that the point at which an iso cost line hits the vertical axis is equal to c/w, where c is the cost represented by that iso-cost line. Thus, the point where iso cost line x hits the vertical axis is equal to $\frac{c(x,w,r)}{w}$, and the point where iso cost line $x+1$ hits the axis is $\frac{c(x+1,w,r)}{w}$. Thus, the difference between the two is equal to

$$\frac{c(x+1,w,r)-c(x,w,r)}{w}$$

or the increase in cost from producing one extra unit of output, divided by the wage rate. Thus, if wages do not change, this gap gives us something that, while not exactly the marginal cost (as the marginal cost is the change in cost from an infinitessimal change in output), is clearly a good approximation to it.

Now look at figure 21. This shows what happens after an increase in r to r' for a particular production function. Recall that an increase in r leads to a steepening of the iso cost lines. Thus, figure 21 now shows 4 iso cost lines, representing the cost minimizing levels for outputs x and $x+1$. for r (the green lines) and r' (the purple lines).. This graph shows two things. First, the cost of producing any level of output is higher for r' than for r, as we would expect. Second, the *marginal* cost is lower for r than it is for r'. This can be seen by noting that the gap between $\frac{c(x,w,r)}{w}$ and

$\frac{c(x+1,w,r)}{w}$ is larger than the gap between $\frac{c(x,w,r')}{w}$ and $\frac{c(x+1,w,r')}{w}$. Thus, this is a case in which an increase in the cost of r increases total cost, decreases marginal cost, and so increases optimal output.

5 When to Pack up and Go Home

Remember that earlier we showed that profits are positive whenever price is higher than average cost. For the most part, our assumptions thus far have insured that this is always the case, as marginal cost has always been above average cost, and as profit maximization occurs at the point at which price is equal to marginal cost, price is definitionally above average cost. However, what about the case where the production function looks like it does in figure 8? Here, marginal cost is initially decreasing and then increasing. Moreover, marginal cost will initially be below average cost, then go above it, as shown in figure 11. In this case it may be that if the firm sets the price equal to marginal cost they make negative profits. Look at Figure 22. This shows two prices, p_1 and p_2. At p_1, marginal cost equals price at x_1, while at p_2, price equals marginal cost at x_2. Does the firm make positive profits in both of these cases? The answer is no. at x_1, price is below average cost, so profits are negative, while at while at p_2 price is above marginal cost, so profits are positive. Thus, if price is p_1, the firm would be better off producing zero.

What is the minimum price at which the firm will produce a positive amount of output? The answer is the minimum average cost point, as shown by p^* in figure 23. At this point, the firm maximize profits by setting price equal to marginal cost at x^* (remember that that the marginal cost line crosses the average cost line at its minimum point) and the firm makes zero profit. At prices below p^* then the firm will make negative profit, while at prices above p^* the firm can make positive profit. Thus, the supply of such a firm will look like it does in figure 24: at prices below p^* the firm will produce zero. At prices above p^* the firm will produce at the point at which price equals marginal cost.

This analysis has so far assumed that a firm who produces zero faces zero costs. While this may be a good long run assumption (the firm can always shut down), it may not be such a good assumption in the short run. For example, if a firm has signed a long term lease on a factory, then they will have to pay rent on that factory, even if they produce no output. In such a case, it will be handy to split the cost of the firm into two portions, a fixed cost F, which we define as the cost of producing zero output, or $c(0)$, and the variable cost of producing x, which we will indicate as $c_v(x)$, which we define as $c_v(x) = c(x) - c(0)$, (from now on we will not write down that c and c_v depend on w and x, but you should remember that this is the case). Thus, the cost function is now

$$\begin{aligned} c(x) &= c(x) - c(0) + c(0) \\ &= c_v(x) + F \end{aligned}$$

and the profit function

$$\pi(x) = p_x x - c_v(x) - F$$

Note that the addition of the fixed costs does not change the conditions for an interior solution, as this occurs when

$$\begin{aligned} \frac{\partial \pi}{\partial x} &= p_x - \frac{\partial c_v(x)}{\partial x} = 0 \\ &\Rightarrow p_x = \frac{\partial c_v(x)}{\partial x} \end{aligned}$$

However, now the firm may be better off producing *even if prices are below average cost.* To see this, note that the firm is better off producing at the point of tangency if the profits for doing so are higher than the profits of producing at zero. But now, the profits for producing at zero are $p_x 0 - c_v(0) - F = -F$. Thus, the firm will want to carry on producing if

$$\begin{aligned} p_x x - c_v(x) - F &\geq -F \\ p_x x - c_v(x) &\geq 0 \\ &\Rightarrow p_x \geq \frac{c_v(x)}{x} \end{aligned}$$

Note that $\frac{c_v(x)}{x}$ is the average *variable* cost of producing at x, which we will denote as $AVC(x)$. Note also that average variable cost is always below average cost, as

$$\begin{aligned} AC(x) &= \frac{c_v(x) + F}{x} \\ &= \frac{c_v(x)}{x} + \frac{F}{x} \\ &= AVC(x) + \frac{F}{x} \end{aligned}$$

This situation is depicted in figure 25: The firm will keep producing as long as prices are above the minimum level of average variable cost, as shown by p^{**}. (as you will see in the homework, the marginal cost curve also crosses the average variable cost curve at the bottom).

This gives us the following theorem

Theorem 1 *Let $p^* = \min_{x \in X} AVC(x)$. Then a firm will produce a positive amount of output if $p > p^*$ and produce zero if $p < p^*$. If $p = p^*$ then they are indifferent between producing either 0 or the output at which $MCx) = p^*$.*

6 Producer Surplus

We have now developed the tools necessary to figure out how much output a firm will produce for a particular price by solving its optimization problem. When in consumer theory, we decided that we could gain some information about how good an outcome was for the consumer by looking at the area under the demand curve. Can we say anything similar about the supply curve? The answer to this, it turns out is yes.

To see this, we will return to the case where we assume that marginal costs are always rising, and there are no fixed costs. In this case, the supply curve will not 'jump' as it did in the examples in section 5, and so at every point of thee supply curve, price will equal marginal cost. Thus supply curve will look like it does in figure.26.

It turns out that the area under the supply curve up to an amount x^* tells us the cost that the firm faces in producing x^*. To see, let $p(x)$ be the price associated with the output level x by the supply curve, and remember that along the supply curve

$$p(x) = MC(x) = \frac{\partial c(x)}{\partial x}$$

Thus, integrating along the supply curve gives us

$$\begin{aligned}
& \int_0^{x^*} p_x(x)dx \\
= & \int_0^{x^*} \frac{\partial c(x)}{\partial x} dx \\
= & [c(x)]_0^{x^*} \\
= & c(x^*) - c(0) \\
= & c(x^*)
\end{aligned}$$

where the last line comes from the fact that we assumed no fixed costs. Thus the area under the supply curve equals the total cost of producing x^*. However, the total revenue of producing an amount x^* at price p^* is just equal to x^*p^*, or the area under the price line p^* up to x^*. As profit equals revenue minus cost, the area between the price line and the supply curve must be equal to profit, or as we sometimes call it, producer surplus. This is shown in figure 27.

Interestingly, as you will show for homework, this is also true for the cases that we covered is section 5, so in general we can think of the area between the supply curve and the price line as being equal to profit

7 Industry Supply

So far we have thought about the case where there is only one firm in the economy. However, this seems to be a bit of a strong assumption. What would the world look like if we had (say) two firms? In particular, what would the supply function look like?

To make thinks simple, lets assume that each firm has the same technology. This means the profit maximizing output of each firm is identical at each price, Lets say that $S_1(p)$ is the amount that firm 1 would produce at price p, and $S_2(p)$ be the amount that firm 2 would produce. Then the total industry supply at price p is given by

$$S(p) = S_1(p) + S_2(p) = 2S_1(p)$$

This is shown in figure 28 for the case where the supply curve does not jump. As you can see, the industry supply curve will get flatter than the individual firm supply curve. As we add more firms, the supply curve will get flatter still.

What about the case in which supply 'jumps'? It is still the case that for prices below p^* (the lowest average cost obtainable by the firm), neither firm will want to produce anything. Above this point, they will want to produce at the point at which their price equal marginal cost. Thus, the industry curve will look as it does in figure 29.

A natural question is whether the idea of producer surplus carries over to the case in which the supply curve is industry supply, rather than firm supply. The answer, it turns out, is yes, as you will see in the homework

A more interesting question is what might determine the number of firms in the economy. In general, economists like to assume that there is *free entry* into a market - that is there is nothing to stop new firms entering a market (you should think about whether you think this is a good assumption). When will new firms choose to do so? The answer to this question is usually 'a firm will enter the industry if it can make positive profits by doing so'. Initially, this can seem like a bit of an odd response:if a firm could only make very small profits by entering the industry, why would they bother. The peculiarity here is in how economists define costs: in calculating costs, we think not just about how much we pay workers, capital, etc, but also included in there is the cost of the *entrepreneur,* or whoever it is that is running the firm. The cost of the entrepreneur is calculated as their *opportunity cost,* or what they would get if there were doing something else. Thus, if a firm is making profits in an economic sense, it means that they are making more than the entrepreneur could make doing anything else, thus an entrepreneur would like to run such a firm, even if the 'profits' are small, as long as they are positive.

So if we assume that there is free entry, can we figure out how many firms an industry can support? Let us first consider the case where the supply curve 'jumps'. This is the case shown in figure 30. Here we show the sum of the supply curves for different numbers of firms, and two demand curves, D_1 and D_2. Looking first at D_1, how many firms could this demand curve support? Well, it could clearly support 1 firm: if there were only one firm in the industry, prices would be p_1 and that firm would produce x_1. It could also support 2 firms, as here the price would be p_2 and each firm would produce $\frac{x_2}{2}$, which is greater that x^*, as $x_2 > 2x^*$. However, it could not support three firms. The minimum amount that needs to be supplied to support 3 firms is 3X*, but in order for demand to be at this level, the price would have to be below p^*. Similarly, D_2 can support 3 but not 4 firms.

This means, with free entry and exit, the supply curve looks like it does in figure 31 - approximately a flat line at p^*, the minimum price at which firms will produce.

8 Partial Equilibrium

With all the work we have done in the course up until now, we are finally back to where you were in econ 11: We have a downward sloping demand curve (which we derived in the section on consumer theory) and an upward sloping supply curve (which we derived above). However, you should not belittle the work we have done: rather than just being arbitrary lines drawn on the board, we now have a proper foundation for the supply and demand curves, and a much better understanding of where these curves come from.

We are now going to use these curves to do some **partial equilibrium** analysis. In practice, this means using the supply and demand curves to analyze the effects of various policies, such as taxes and quotas. The reason we call this *partial* equilibrium analysis is that we are looking at one market at a time. Think back to when we did equilibrium in the exchange economy with two goods: figs and brandy. Changes in one market would implicitly effect what happened in the other market. Moreover, the welfare of a agents would depend on what happened in both markets. Analysis that takes this into account is called **general equilibrium.** Here we are abstracting from that, by concentration only on one market (say the market for figs). Thus, we are implicitly assuming that the market we are interested in can be examined in isolation from the rest of the economy.

With that caveat in mind, partial equilibrium analysis can still be extremely useful. The basic idea is that the market is in equilibrium when prices are such that supply (derived from the firms's optimization problem) is equal to demand (as derived from the consumer's optimization problem). Thus, figure 32 gives an example of equilibrium in the market for figs.

We can use these diagrams to analyze what would happen to equilibrium prices and quantities in response to changes in supply and demand conditions. First, let us consider what happens in the face of a supply 'shock'. By this I mean a change in conditions which means that firms demand a higher price to produce any given level of output, or an upward shift in the supply curve. Typically, people will not be very precise about what it is that causes a supply shock. For example, one could say that there was a blight on figs trees, reducing the number of fig trees available. But does this necessarily mean a shift up in the supply curve? Remember that the supply curve is determined by the *marginal* cost of producing figs, not the *total* cost. You should think very carefully about what sort of changes do imply a shift in the supply curve.

Figure 33 shows the effect of a negative supply shock (i.e. an increase in the marginal cost of production - let's say due to a blight): this is defined as a shift in the supply curve upwards from the pre-blight to the post blight line. It should be obvious that equilibrium prices and quantities shift from x_1, p_1 to x_2, p_2, leading to an *increase* in the price of figs and a *decrease* in the quantity of figs produced and consumed.

In contrast, figure 34 shows the effect of a demand shock. This is a shock that effects the amount that people are prepared to pay for any amount of figs - Figure 34 shows the effect of a positive demand shock, or an increase in the price that the consumer is prepared to pay for any level of figs. We will think of this as resulting from a fig festival, when it is traditional for people to eat figs, but once again you should be very careful about what type of change actually will lead to a shift in the demand curve. A positive demand shock will lead to an *increase* in the price of figs and an *increase* in the quantity of figs demanded.

We can also use this diagram to do welfare analysis, using the concept of consumer and producer surplus (are remembering all the caveats we made when constructing these concepts). Remember that the consumer surplus is given by the area between the demand function and the price line, while producer surplus is the area between the supply curve and price line as shown in figure 35. The total of consumer and producer surplus we refer to as **total surplus.**

We can use the concept of total surplus to analyze the effect of various policies. First, we are going to think about the effect of an export ban.

Example 3 (Trade Ban) *For this example, we are going to thing about the market for miniature American flags. We will assume that the demand curve for a miniature American flags within the US is given by*

$$D(p) = 20 - p$$

while the supply function is given by

$$S(p) = 3p$$

Note that, while we are now just writing down demand and supply functions seemingly arbitrarily, we should still think of these as resulting from the optimal decisions of producers and consumers.

We are also going to assume that there is a global market for miniature American flags: On the global market, consumers can buy as many flags as they want at \$4, and producers can sell as many flags as they want at \$4. thus, at \$4, global demand and supply is infinitely elastic. This is a common trick that economists use. The justification is that the global market is 'large' relative to the domestic market, so the amounts demanded or supplied by the US will not have a great effect on global prices.

How do we analyze this situation using our diagrams. Well, let's first figure out what demand and supply would be when the price is \$4.

$$\begin{aligned} D(4) &= 20 - 4 = 16 \\ S(4) &= 12 \end{aligned}$$

This means, if domestic consumers are allowed to buy on the global market, then the price in the domestic market will also be \$4, domestic demand will be 16 flags, domestic supply will be 12 flags, and the US will import 4 flags, as shown in figure 36. The surplus under this regime is shown in figure 37, which we can calculate explicitly. Producer surplus is given by

$$\begin{aligned} PS &= \frac{1}{2} 12 \times 4 = 24 \\ CS &= \frac{1}{2} 16 \times 16 = 128 \end{aligned}$$

Where these are just the areas of the respective triangles. Total surplus is thus 152.

Now let us analyze the effect of a protectionist policy. A minister decides that, not only is it unpatriotic to import miniature American flags, it is also harming local workers by suppressing wages in the miniature flag making industry. Therefore imports should be banned. Following implementation of the policy, (partial) equilibrium is reached when prices are such that domestic demand equals domestic supply, (a situation sometimes called autarky). What is this price? We can solve for it by setting supply and demand equal:

$$\begin{aligned} D(p) &= 20 - p = 3p = S(p) \\ 20 &= 4p \\ p &= 5 \end{aligned}$$

And at this price $S(5) = D(5) = 15$*. This situation is shown in figure 38. We can calculate* *surplus in this case as*

$$
\begin{aligned}
PS &= \frac{1}{2}15 \times 5 = \frac{75}{2} \\
CS &= \frac{1}{2}15 \times 15 = \frac{225}{2}
\end{aligned}
$$

The policy therefore increases the surplus of producers but reduces the surplus of consumers. Total surplus is therefore $\frac{300}{2} = 150$. *Thus the policy has lowered total surplus. We can see this from figure 39, which shows that total surplus falls by an area equivalent to the blue triangle. This is called* ***deadweight loss****, and it should be clear that if the supply curve is upward sloping and the demand curve is downward sloping we will always have a deadweight loss from such a policy.*

Arguments like the above are usually taken by economists to suggest that we should not make such a change, as there is a loss of efficiency. Because the fall in consumer surplus is larger that the rise in producer surplus, consumers could afford to compensate producers in order to be allowed to keep buying at the world price. However, I am not convinced that this is a particularly good argument. Apart from the problems that we have discussed with consumer and producer surplus, it is also not the case that it is Pareto dominant to allow trading on the global market. Sure, there may be a Pareto dominant outcome in which consumers are allowed to trade on the world market, and then compensate producers, but this seems to very rarely happen in practice (think second welfare theorem). In fact, one of the effects of globalization seems to have been to bid down the wages of people who work in export-competing parts of the economy (e.g. manufacturing), and it is not often the case that such people are compensated for their losses (though note that, as consumers as well, these people also get to benefit from the lower costs inherent in the world market).

A second policy we can analyze within this framework is that of a tax

Example 4 (Sales Tax) *Imagine that there is no world market for miniature american flags, so the economy is now at autarky. Some damned American hating socialist decides to implement a $4 tax on each flag sold. What does the (partial) equilibrium look like now? Well, let* p_d *be the price that the consumer pays, and* p_s *be the price that the producer receives. In equilibrium,*

$$D(p_d) = S(p_s)$$

We also know that the difference between these two prices is equal to the tax paid, and so $p_d = p_s + 4$. *This means we can find equilibrium:*

$$\begin{aligned} D(p_d) &= S(p_s) \\ D(p_s + 4) &= S(p_s) \\ 20 - p_s - 4 &= 3p_s \\ 16 &= 4p_s \\ p_s &= 4 \end{aligned}$$

And so $p_d = 8$, *and* $D(p_d) = S(p_s) = 12$. *This is the situation shown in figure 40.*

We can also analyze the welfare effect of this tax. Consumer surplus and producer surplus obviously fall a lot

$$\begin{aligned} PS &= \frac{1}{2} 12 \times 4 = 24 \\ CS &= \frac{1}{2} 12 \times 12 = 72 \end{aligned}$$

However, the picture is not quite as bleak as that,as the government receives some tax revenue. This is equal to the tax per item times the number of items sold

$$revenue = 12 \times 4 = 48$$

Thus, 'social surplus' (i.e. consumer surplus+producer surplus +tax revenue) is 72+24+48=144. The deadweight loss of the tax is 6, as illustrated in figure 40.

In this case, the deadweight loss of the tax comes about due to inefficiency: The tax drives a 'wedge' between the price that people are willing to pay for a good, and the price that firms would accept to produce the same good. This is a point that we will come back to.

9 General Equilibrium

Partial equilibrium is certainly a useful concept, but there are reasons why, in the Edgeworth box, we looked at both markets at the same time. There are important results that only can be derived by thinking of the economy as a whole - the situation we call general equilibrium. In particular, when thinking about an economy in which there were only consumers we derived the rather extraordinary result that any competitive equilibrium is pareto optimal. I promised then that we would show this also to be the case when we introduced production, and now it is time to come good on that promise. However, as is typical, we are going to do this in a weird and roundabout way. I am first going to give an example, and then I am going to prove the statement directly.

Worse still, the example I am going to give is going to seem *really* weird. It is what we call the **Robinson Crusoe Economy**

9.1 The Robinson Crusoe Economy

As we all know, Robinson Crusoe was stranded on an island, and has to subsist on local materials. What is less well know (except to economists) is that in doing so, he started to behave in a very odd manner - in fact, he started to suffer from a split personality. In particular, he split himself into Robinson Crusoe the worker and Robinson Crusoe Inc., a firm. The former was a simple man, who had preferences over a consumption good (coconuts) and leisure. The coconuts he bought on the local market, while he sold his labor to the local firm - Robinson Crusoe Inc. The latter was a dynamic, thrusting firm which bought labor off Robinson Crusoe (the man), used this labor to get coconuts, and sold these coconuts to Robinson Crusoe (the man) in order to maximize profits. These profits go to the sole owner of Robinson Crusoe Inc., a man called Robinson Crusoe.

Why on earth am I telling you this ridiculous story? The reason is that this is a way of analyzing the simplest production economy - one in which there is one consumer (Robinson Crusoe, henceforth RC), one firm (Robinson Crusoe Inc., henceforth RCI), one input to production (labor) and one output (coconuts). What we are going to show is that, in this economy, the competitive equilibrium is pareto optimal, in the sense that it will lead to a combination of labor and coconuts that will maximize RC's preferences.

In order to see this, we have to figure out what such a combination looks like. Doing so involves solving what we call the 'planners problem', or imagining that we are a benevolent social planner who get to choose a bundle of coconuts and leisure in order to maximize RC's preferences. In other words, the planner's problem asks is the following

Choose: *an output level c and levels of input level l*

In order to maximize: *RC's preferences* $u(c, 24 - l)$

Subject to: *the production function* $c = f(l)$

Where c is the number of coconuts that RC consumes, and l is the amount of labor RC spends looking for coconuts.

In order to solve this problem, we will start by graphing the constraint. This comes from the production function: We cannot give RC more coconuts than can be produced given the labor he provides. We will assume that the production function $f(l)$ has all the nice properties that we have so far assumed, and so it looks like it does in figure 41. Thus, we know that we have to pick a point somewhere on this line: this is the combination of labor and coconuts that we have to choose from

However, look at the axis on this graph: these measure labor and coconuts. But these are the same things that RC has preferences over. Thus, we can graph RC's preferences on the same graph. What are they going to look like? Well, assuming that they are 'nice' (i.e. monotonic in leisure and coconuts and convex), the indifference curves will look like they do in figure 42 -the 'mirror image' of standard preferences (as the bottom axis shows labor or 24-minus leisure, and so is a 'bad' with respect to RC's preferences).

By now, we should be very used to figuring out how to solve this sort of thing. We want to locate the point on the production function that is going to allow us to get on the highest of RC's indifference curves. This maximum is going to come at the point where the indifference curves are tangential to the constraint - which in this case is the production function, as shown in figure 43. And what is true at this point? The slope of the production function is the same as the slope of the indifference curve. The slope of the production function is just the marginal product of labor, while the slope of the indifference curve is the marginal rate of substitution between leisure and coconuts.

Another way to see the same result is by rewriting the planners problem above as an unconstrained optimization problem by substituting in of the production function

Choose: *a level of input level l*

In order to maximize: *RC's preferences* $u(f(l), 24 - l)$

Now, in order to find the optimum, we simply differentiate the utility function with respect to leisure, giving

$$\begin{aligned}\frac{\partial u}{\partial l} &= \frac{\partial u}{\partial c}\frac{\partial f}{\partial l} - \frac{\partial u}{\partial l} \\ \Rightarrow \quad \frac{\partial f}{\partial l} &= \frac{\frac{\partial u}{\partial l}}{\frac{\partial u}{\partial c}} = MRS_{l,c}\end{aligned}$$

Thus, the 'pareto optimum' in this economy occurs at the (unique)[1] point on the production function at which marginal product equals the marginal rate of substitution.

A moment's thought should tell us that this makes sense. Marginal product is the rate at which RCI can convert labor into coconuts. The MRS measures the rate at which coconuts can be traded for leisure while keeping RC indifferent. If, for example, the marginal product was higher than the MRS, then we could make RC happier by getting him to provide one extra unit of labor. The increase in coconuts that RC gets as a result (measured by the marginal product) would more than offset him for the loss of utility as measured by the MRS.

Now we know what the pareto optimum looks like. What does the competitive equilibrium look like? First, we have to define what a competitive equilibrium is in an economy with production. In order to do so, start by recalling how we defined a competitive equilibrium in an economy with only consumers:

Definition 2 (Equilibrium with consumers only) *An equilibrium is a consumption bundle for each consumer and a price p such that*

1. *The allocation is feasible*
2. *The consumption bundles solve the consumers optimization problem, given the price and initial allocations*

In other words, we figured out an amount of each good to give to each person such that the total amount we allocated was equal to the amount of that good in the economy (feasibility), and that at the equilibrium price, those are the amounts that the consumers would have chosen to maximize their preferences (optimization). Put another way, we found a set of prices such that demand at those prices equalled supply.

[1] Question: What assumption have we made that ensures that this point is unique?

We can modify this definition to allow for an economy with producers as well as consumers

Definition 3 (Equilibrium with consumers and firms) *An equilibrium is a consumption bundle for each consumer, a production decision for each firm and a set of prices p such that*

1. *The allocation is feasible*
2. *The consumption bundles solve the consumers optimization problem and the production decision of each firm maximizes profits, given prices*

In other words, we tell each firm exactly what to make, using what inputs, and tell each consumer what to consume, **and** post a set of prices such that feasibility is satisfied, and the consumption bundles and production decisions are exactly what the consumers and firms would have chosen to do, given those prices.

One question you should ask is: what does feasibility mean when we have production? Well think of our Robinson Crusoe economy. Here there are two 'goods': labor and coconuts. The only supply of labor comes from RC, while the only demand for labor comes from RCI. In contrast, the only demand for coconuts comes from RC, while the only supply comes from RCI. Thus, feasibility means that the amount of labor used by RCI is equal to the amount of labor supplied by RC, while the amount of coconuts produced by RCI is equal to the amount eaten by RC. You can generalize the notion of feasibility to a case where there are more firms, more consumers, more goods and more inputs: feasibility means that, for each input, total amount used by all firms is equal to the total amount supplied by all households, while for each good, total amount consumed by all the consumers is equal to the total amount supplied by all the firms.

Returning to our Robinson Crusoe economy, another way to think about the an equilibrium is that we are looking for a set of prices such that two things are true:

1. The demand for labor by RCI is equal to the supply of labor from RC.
2. The demand of coconuts from RC is equal to the supply of coconuts from RCI

What will the competitive output look like? Lets start by thinking about the behavior of a firm when the wage rate is w (we will normalize the price of coconuts to 1). We know that the firm will choose to produce at the point where the iso profit line is tangential to the production function. Remember that the iso profit line is given by

$$c = \pi + wl$$

So the firm will choose to produce at the point where marginal product is equal to w, and the profit that the firm earns is given by the intercept of the iso cost line with the vertical axis. We will call the associated demand for labor, supply of coconuts, and profits $l_f(w)$, $c_f(w)$ and $\pi(w)$ respectively. This situation is shown in figure 44. As we know, the marginal product has to equal the wage rate at this point, so:

$$\frac{\partial f(l(w))}{\partial l} = w$$

Note also that the point at which the tangential iso-profit line hits the vertical axis is the profit that the firm makes.

How about the consumer, RC? First of all, what is RC's income? This comes from two sources: the wages that he gets from selling labor ($l(w)$), and the profits he gets from owning RCI ($\pi(w)$).If he is to consume all his income, then the number of coconuts that he buys has to be equal to this income, so his constraint is

$$c = \pi(w) + wl$$

Does this look familiar? It should, as this is exactly the same as the equation of the iso-cost line going through the firm's profit maximizing output. Amazing!

Given this constraint, what will RC choose to consume? As usual, he will choose to consume at the point at which the budget constraint is tangential to his indifference curves, as shown in figure.45. We will denote the demand for coconuts and supply of labor that RC would choose as $c_c(w)$ and $l_c(w)$ respectively. And at this point, we know the marginal rate of substitution between leisure and coconuts is equal to the slope of the budget constraint, so

$$w = MRS_{l_c(w),c_c(w)}$$

So does the wage rate w that we have chosen here support an equilibrium? The answer is no, as shown in figure 46.At the wage rate we have chosen, then RC's demand for coconuts is larger than RCI' supply ($c_c(w) > c_f(w)$) and RC's supply of labor is greater than RCI's demand ($l_c(w) < l_f(w)$).

For the wage rate to support an equilibrium, it has to look like it does in figure 47. Here, wage rate w^* is such that where $c_c(w^*) = c_f(w^*)$ and $l_c(w^*) = l_f(w^*)$, or the labor that RC wants to supply is equal to the labor that RCI demands, and the coconuts that RC demands are equal to the coconuts that RCI supplies.

But note that something else is true about this point. At the equilibrium, we know that the slope of the indifference curve is equal to the wage rate

$$w^* = MRS_{l_c(w^*),c_c(w^*)}$$

and we know that wage rate is equal to marginal productivity:

$$\frac{\partial f(l(w^*))}{\partial l} = w^*$$

Putting these two together gives

$$\frac{\partial f(l(w^*))}{\partial l} = MRS_{l_c(w^*),c_c(w^*)}$$

But (gasp!) this is exactly the same condition as we had for the pareto optimum! The competitive equilibrium is the one that maximizes RC's utility! The first fundamental theorem goes through! We're Saved!

What is driving this extraordinary result? It turns out that once again, prices are acting as devices that equalize marginal rates, and this is the condition that we need to ensure optimality: When behaving competitively, the firm sets the rate at which it can convert labor to coconuts equal to the wage rate. Similarly, the consumer sets the rate at which they convert labor into coconuts at the wage rate. Thus, the equilibrium equilibrates these two marginal rates, which is what we need for an equilibrium.

Given that the Robinson Crusoe economy is even more specific than some others that we have looked at, you might be a bit suspicious about this result . Is this true if we have more than one consumer and more than one firm? Have I pulled a fast one by focussing in Mr. Crusoe? The answer is no: Production economies generally satisfy the first fundamental welfare theorem, as we now show. For simplicity we will think of the case where there are two goods in the economy x and y, and two inputs into production k and l. The game we are going to play is look at a competitive allocation in the economy, which consists of the following objects:

x_1, y_1, l_1, k_1 the good consumed and factors of production sold by person 1

x_2, y_2, l_2, k_2 the good consumed and factors of production sold by person 2

x_f, l_x,k_x, the amount produced and the inputs used by the firm producing x

y_f, l_y,k_y, the amount produced and the inputs used by the firm producing y

and consider another allocation (which we will indicate by putting stars in front of everything) that pareto dominates the competitive allocation. We will show that this new allocation must be infeasible.

Let's start off by thinking of the budget constraint of individual 1 at competitive equilibrium:

$$p_x x_1 + p_y y_1 = w l_1 + r k_1 + \alpha \pi_x + \beta \pi_y$$

x_1 and y_1 are the amount of x and y that 1 consumes, while l_1 and k_1 are the amount of labor and capital that they produce. α is person 1's share of the firm that produces x, while β is their share of the firm that produces y. Now, note that, if there was another bundle of inputs and outputs that 1 could supply x_1^*, y_1^* l_1^*, k_1^* that 1 would prefer, then they would not be able to afford it, so it would have to be the case that

$$\begin{aligned} & p_x x_1 + p_y y_1 - w l_1 - r k_1 \\ > \; & p_x x_1^* + p_y y_1^* - w l_1^* - r k_1^* \end{aligned}$$

Therefore, if we add across the two individuals in the economy, we get

$$p_x(x_1 + x_2) + p_y(y_1 + y_2) - w(l_1 + l_2) - r(k_1 + k_2) = \pi_x + \pi_y$$

And, for any pareto improving bundle, it must be that

$$\begin{aligned} & p_x(x_1^* + x_2^*) + p_y(y_1^* + y_2^*) - w(l_1^* + l_2^*) - r(k_1^* + k_2^*) \\ > \; & p_x(x_1 + x_2) + p_y(y_1 + y_2) - w(l_1 + l_2) - r(k_1 + k_2) \end{aligned}$$

Now what about the profits of the two firms? These are given by

$$\begin{aligned} \pi_x &= p_x x_f - w l_x - r k_x \\ \pi_y &= p_y y_f - w l_y - r k_y \end{aligned}$$

Again, as the firms are maximizing profits, then it has to be that these profits are higher than under the alternative plan, so

$$\begin{aligned} p_x x_f - w l_x - r k_x &> p_x x_f^* - w l_x^* - r k_x^* \\ p_y y_f - w l_y - r k_y &> p_y y_f^* - w l_y^* - r k_y^* \end{aligned}$$

Adding these two together tells us that

$$\begin{aligned} & p_x x_f + p_y y_f - w(l_x + l_y) - r(k_x + k_y) \\ > \; & p_x x_f^* + p_y y_f^* - w(l_x^* + l_y^*) - r(k_x^* + k_y^*) \end{aligned}$$

As the competitive allocation was feasible, we know that

$$\begin{aligned} (x_1 + x_2) &= x_f \\ (y_1 + y_2) &= y_f \\ (l_1 + l_2) &= (l_x + l_y) \\ (k_1 + k_2) &= (k_x + k_y) \end{aligned}$$

and so

$$\begin{aligned} & p_x(x_1 + x_2) + p_y(y_1 + y_2) - w(l_1 + l_2) - r(k_1 + k_2) \\ = \; & p_x x_f + p_y y_f - w(l_x + l_y) - r(k_x + k_y) \end{aligned}$$

implying that

$$\begin{aligned} & p_x(x_1^* + x_2^*) + p_y(y_1^* + y_2^*) - w(l_1^* + l_2^*) - r(k_1^* + k_2^*) \\ > \; & p_x x_f^* + p_y y_f^* - w(l_x^* + l_y^*) - r(k_x^* + k_y^*) \end{aligned}$$

Rearranging this expression tells us that feasibility must fail, as

$$\begin{aligned} & p_x(x_1^* + x_2^* - x_f^*) \\ & + p_y(y_1^* + y_2^* - y_f^*) \\ & + w(l_1^* + l_2^* - l_x^* + l_y^*) \\ & + r(k_1^* + k_2^* - k_x^* + k_y^*) \\ > \; & 0 \end{aligned}$$

10 Externalities

We are now going to return to a topic which we have touched on before: externalities. We discussed this in the context of equilibrium in the consumer economy, where we described externalities as a circumstance in which the consumption of one good by one person directly affected the utility of another. Now we have firms, we can more generally think of externalities as a case in which the economic activity of one economic agent (firm or consumer) directly affects another. As was the case with the consumer based economy, this can lead to the competitive equilibrium being inefficient.

We are going to start with a classic example: pollution. In particular, we are going to think of two industries: Stan's Tannery and Ian's Aromatherapy Spa. Stan's Tannery is upwind of the spa, thus, if Stan is producing leather, then Ian is affected by the smell. In particular, we are going to assume that Ian can still do aromatherapy, but it costs him more, due to the need for more candles to hide the smell. The more that Stan produces, the more that Ian has to spend. Thus, the profits of the two firms are given by

$$\begin{aligned}\pi_t &= p_t t - c_t(t) \\ \pi_a &= p_a a - c_a(t,a)\end{aligned}$$

Where, the first line is the profits for the tannery and t is the amount the tannery produces, while the second line is the profits for the aromatherapy spa, with a being the amount of aromatherapy that Ian does. What makes this problem non-standard is the fact that $c_a(t,a)$, the cost of aromatherapy, depends on t. We have also assumed that

$$\frac{\partial c_a(t,a)}{\partial t} > 0$$

What output of tanned goods and aromatherapy will maximize producer surplus, in terms of the total profits of the two industries? (I know that isn't the same as welfare, but this is an easy way of making the point). Joint profits are given by

$$\pi_t + \pi_a = p_t t - c_t(t) + p_a a - c_a(t,a)$$

Thus, taking first order conditions gives

$$\begin{aligned}\frac{\partial(\pi_t+\pi_a)}{\partial a} &= p_a - \frac{\partial c_a(t,a)}{\partial a} = 0 \\ &\Rightarrow p_a = \frac{\partial c_a(t,a)}{\partial a}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial(\pi_t+\pi_a)}{\partial t} &= p_t - \frac{\partial c_t(t)}{\partial t} - \frac{\partial c_a(t,a)}{\partial t} = 0 \\ &\Rightarrow p_t = \frac{\partial c_t(t)}{\partial t} + \frac{\partial c_a(t,a)}{\partial t}\end{aligned}$$

Thus, the optimal thing in terms of producer surplus is for tanned goods to be produced up to the point where the price of tanned goods is equal to the marginal cost of producing tanned goods, plus *the marginal effect of producing tanned goods on the cost of aromatherapy.* Assuming that $c_t(t)$, this is a situation depicted in figure 47, with a welfare maximizing output of t^*.

What will Stan *actually* choose to produce? He will, of course, choose the profit maximizing output for his firm, and so set price equal to marginal cost:

$$\begin{aligned}\frac{\partial(\pi_t)}{\partial t} &= p_t - \frac{\partial c_t(t)}{\partial t} = 0 \\ &\Rightarrow p_t = \frac{\partial c_t(t)}{\partial t}\end{aligned}$$

A situation depicted in figure 48. The profit maximizing output is higher that the social optimum, because Stan does not take into account the effect of his tanning on Ian's Aromatherapy. What is to be done? Two possible solutions are

1. Tax the production of leather, as shown in figure 49
2. Force the two firms to merge. Then the single owner will try and maximize total profits, and so take into account the effect of the tannery on the aromatherapy business.