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Intermediate Microeconomics W3211

Lecture 10: Recap
Consumer Problems and Market Equilibrium

Columbia University, Spring 2016
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Introduction

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The Story So Far....

Lecture	Topic
1	Setting up the consumer problem: bundles, budget sets and preferences
2	Preferences, indifference curves and utility
3	Types of preference: Monotonicity, convexity, perfect complements, substitutes and Cobb Douglas
4	Solving the consumer's problem: corner solutions, kinks and tangents
5	Solving the consumer's problem: derivatives and Lagrangians Demand functions and the effect of income
6	Demand functions: own-price changes and the Slutsky Equation; cross price changes and income and substitution effects

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The Story So Far....

Lecture	Topic
7	The endowment economy and the Edgeworth box. Competitive Equilibrium
8	Pareto optimality and the first fundamental theorem of welfare economics
9	The second fundamental theorem of welfare economics Externalities and Coase theorem

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The Story So Far....

- This may seem like a lot of stuff
- However, **a lot of it** has to do with setting up, solving and using the consumer's problem

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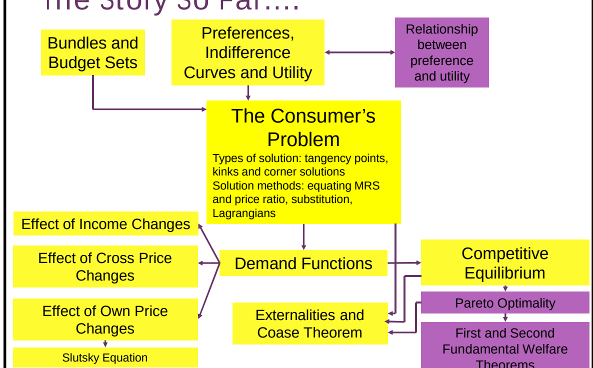
The Story So Far....



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The Story So Far….

- This may seem like a lot of stuff
- However, **a lot of it** has to do with setting up, solving and using the consumer's problem
- If you can figure out how to get very comfortable with doing this, then everything else will come very easily
- You should be able to solve them in your sleep
- Another advantage: we are soon going to move on to the problem of the firm
- Guess what: another constrained optimization problem!
- The tools you have learned to solve the consumer's problem will also be useful in solving the firm's problem

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The Plan for Today

1. Think about solving consumer's problem again (nice and slowly)
2. Go over the income and substitution effect of price changes, and the Slutsky equation
3. Questions (if time)

Solving the Consumer's Problem

A Review

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Solving the Consumer's Problem

- The best way of getting a handle on the consumer's problem is to think about it in terms of pictures
  - Budget Sets
  - Indifference curves

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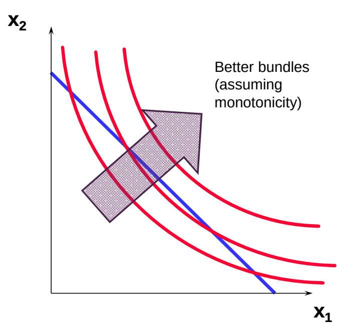
Draw a Picture!



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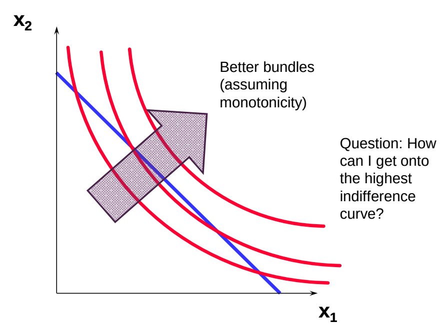
Draw a Picture!



Draw a Picture!

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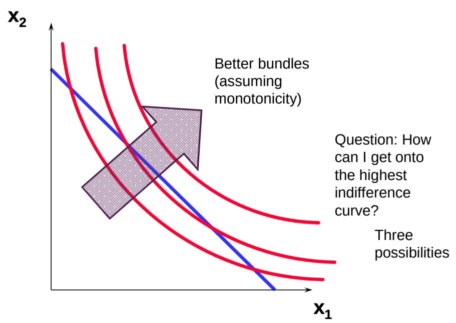

Possibility 1: Tangency Point

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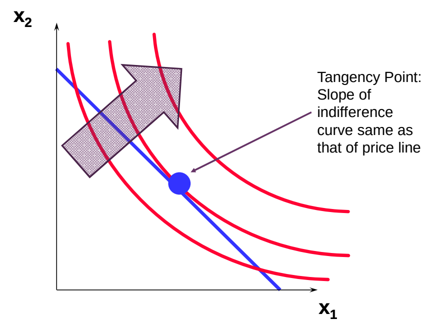

Possibility 2: Corner Solution

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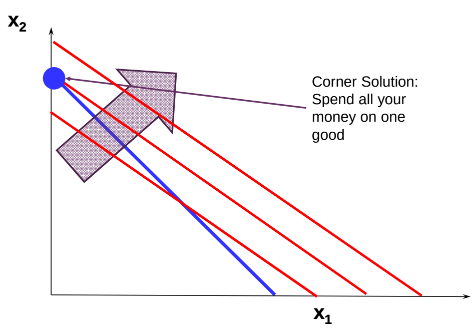

Possibility 3: Kink

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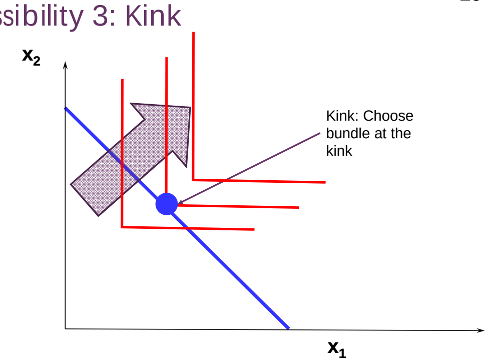

Solving the Consumer's Problem

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- The best way of getting a handle on the consumer's problem is to think about it in terms of pictures
  - Budget Sets
  - Indifference curves
- Try to draw a picture with monotonic preferences at which the optimum doesn't occur at a tangency point, corner or kink
  - You can't!
- Pictures can also be useful if you are finding a problem confusing
  - Sketch indifference curves and budget constraint
  - E.g. in the quiz

Solution to the Quiz

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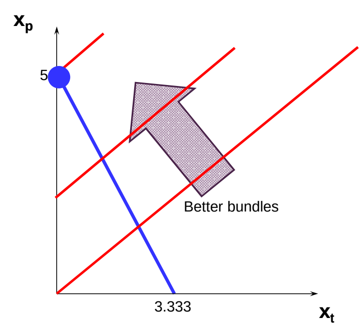

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Solving the Consumer's Problem

- I'm now going to remind you of the recipe that I gave you for solving the consumer problem
- We are going to apply it to a bunch of preferences that have shown up so far
- If you want, you can learn 'by rote' how to work with these preferences
- I don't think this is the optimal thing for you to do
  - I might give you different preferences in the exam!
- Instead, try to use this as a way to get intuition about what the recipe does

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The Recipe

1. Are preferences monotone?
   - If **yes**, then the optimal solution must lie on the budget line
   - If **no** you may have to worry about solutions away from the line
2. Assuming preferences are monotone, there are two possible types of solution
   - **Corner** solutions
   - **Interior** solutions
3. Calculate the utility at each possible corner solution
4. Find all possible interior solutions
   - Points of tangency
   - Kinks
5. Calculate utility at each possible interior solution
6. Compare utilities at **all** possible solutions
7. Select the best

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Finding Points of Tangency

- If you are looking for points of tangency, you have three possibilities:

1. Set MRS equal to the price ratio
   - Lecture 4
2. Substitute using the budget constraint then take derivatives
   - Lecture 5
3. Use Kuhn Tucker (Lagrangians)
   - Lecture 5

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Different Types of Preference

Preferences	Utility fn	Corner Soln?	Kink?	Tangent?
Cobb Douglas				
Perfect Sub				
Perfect Comp				
Quasi Linear				
Concave				

- Concentrating here on monotonic preferences

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Cobb Douglas Preferences

- Utility function: for $a > 0, b > 0$

$$u(x_1, x_2) = x_1^a x_2^b$$

Or $u(x_1, x_2) = a\ln x_1 + b\ln x_2$

- Indifference curve:

$$x_2 = \frac{u^{\frac{1}{b}}}{x_1^{a/b}}$$

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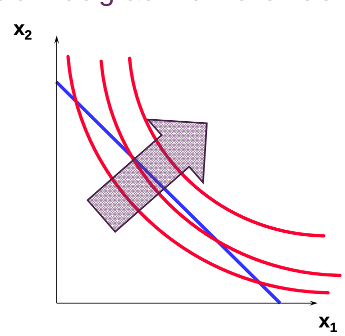
Cobb Douglas Indifference Curves



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Cobb Douglas Preferences

- Can there be a corner solution?
  - No!
  - Utility is zero if one good is consumed at zero
  - Can always do better by consuming some of each good
  - Also, note that MRS goes to infinity as amount of good 1 goes to zero
    - Would only trade an infinite amount of good 1 for an additional unit of good 2
- Can there be a kink?
  - No! Indifference curve is smooth!
- Solution must be a point of tangency

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Cobb Douglas Preferences

$$u(x_1, x_2) = x_1^a x_2^b$$

- Taking derivatives gives

$$MU_1 = \frac{\partial u}{\partial x_1} = a x_1^{a-1} x_2^b$$
$$MU_2 = \frac{\partial u}{\partial x_2} = b x_1^a x_2^{b-1}$$

- And so MRS is

$$\frac{\partial u}{\partial x_1} \Big/ \frac{\partial u}{\partial x_2} = \frac{a x_2}{b x_1}$$

- Setting this equal to the price ratio gives

$\frac{a x_2}{b x_1} = \frac{p_1}{p_2}$ or $x_2 = \frac{b p_1}{a p_2} x_1$

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Cobb Douglas Preferences

$$x_2 = \frac{b p_1}{a p_2} x_1$$

- Substituting into the budget constraint gives

$p_1 x_1 + p_2 x_2 = I$ or

$$p_1 x_1 + p_2 \frac{b p_1}{a p_2} x_1 = I$$

- And so

$$x_1 = \frac{I}{p_1} \frac{a}{a+b}$$

$$x_2 = \frac{I}{p_2} \frac{b}{a+b}$$

- Question: How does the fraction of income spent on each good change with price?

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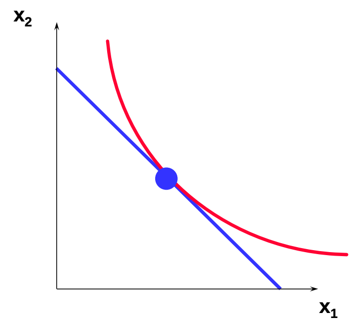
Cobb Douglas Indifference Curves



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Different Types of Preference

Preferences	Utility fn	Corner Soln?	Kink?	Tangent?
Cobb Douglas	$x_1^a x_2^b$ $a \ln x_1 + b \ln x_2$	Never	Never	Always
Perfect Sub				
Perfect Comp				
Quasi Linear				
Concave				

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Perfect Substitutes

- Utility function: for $a > 0$, $b > 0$

$$u(x_1, x_2) = a x_1 + b x_2$$

- Indifference curve:

$$x_2 = \frac{u}{b} - \frac{a}{b} x_1$$

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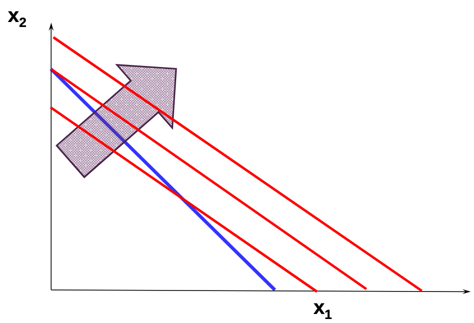
Perfect Substitutes



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Perfect Substitutes

- Is there a kink?
  - No – indifference curve is smooth
- Can the solution be a point of tangency?
  - Remember the indifference curve is given by $x_2 = \frac{u}{a} - \frac{a}{b}x_1$
  - Slope doesn't vary with x_1, it is always equal to $-\frac{a}{b}$
  - There will only be a tangency point in the (very special case) that $\frac{p_1}{p_2} = \frac{a}{b}$
- Generally, solution will be a corner solution
- Which corner? Depends on whether the budget line or the indifference curve is steeper

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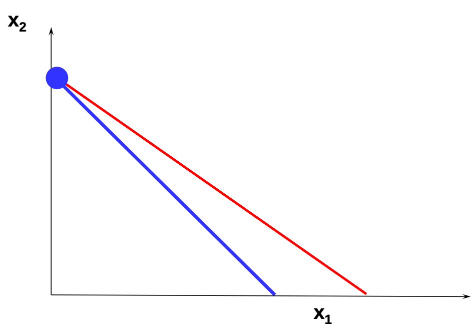
Perfect Substitutes



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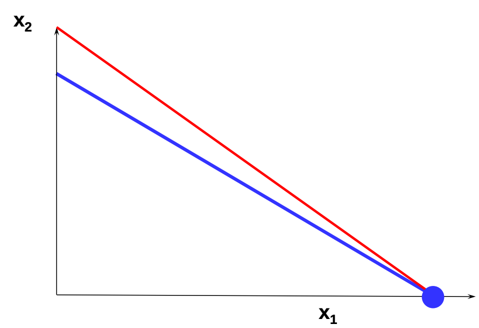
Perfect Substitutes



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Perfect Substitutes

- Substituting into the budget constraint
- If $\frac{p_1}{p_2} > \frac{a}{b}$

$$x_1 = 0, x_2 = \frac{I}{p_2}$$

- If $\frac{p_1}{p_2} < \frac{a}{b}$

$$x_1 = \frac{I}{p_1}, x_2 = 0$$

- What if $\frac{p_1}{p_2} = \frac{a}{b}$?

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Perfect Substitutes

- What if $\frac{p_1}{p_2} = \frac{a}{b}$?
- If consume x_1, then can also consume $\frac{I - p_1 x_1}{p_2}$ units of x_2
- Utility given by

$$ax_1 + b\frac{I - p_1 x_1}{p_2}$$
$$= bI + \left(a - b\frac{p_1}{p_2}\right)x_1$$
$$= bI$$

- Utility doesn't depend on the choice of x_1
- As long as all money is spent, any split of x_1 and x_2 is optimal
- (Note, corner solutions are still optimal)

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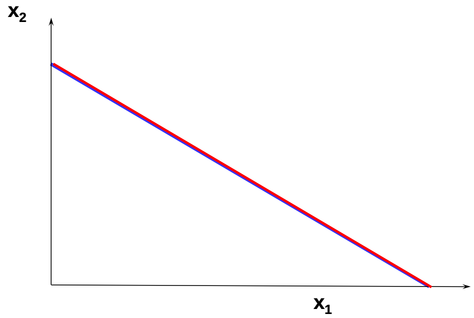
Perfect Substitutes



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Different Types of Preference

Preferences	Utility fn	Corner Soln?	Kink?	Tangent?
Cobb Douglas	$x_1^a x_2^b$ or $a\ln x_1 + b\ln x_2$	Never	Never	Always
Perfect Sub	$a x_1 + b x_2$	Always	Never	Rarely
Perfect Comp				
Quasi Linear				
Concave				

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Perfect Complements

- Utility function: for $a > 0$, $b > 0$

$$u(x_1, x_2) = \min(ax_1, bx_2)$$

- Indifference curve: L-shaped with kink at $\frac{a}{b}$

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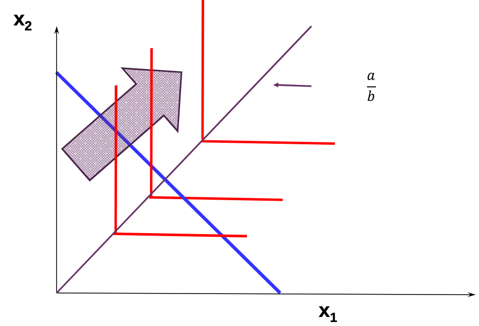
Perfect Complements



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Perfect Complements

- Can there be a corner solution?
  - No!
  - Utility is zero if one good is consumed at zero
  - Can always do better by consuming some of each good
- Can there be a point of tangency?
  - No!
  - Slope of indifference curve is zero or infinity
- Solution has to be at the kink

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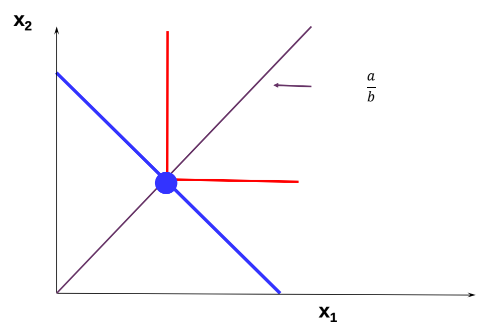
Perfect Complements



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Perfect Complements

- Kink occurs when $ax_1 = bx_2$
- Implies $x_2 = \frac{a}{b}x_1$
- Subbing into budget constraint gives

$$p_1x_1 + p_2x_2 = I \text{ or}$$

$$p_1x_1 + p_2\frac{a}{b}x_1 = I$$

- And so

$$x_1 = \mathrm{I}\frac{b}{bp_1 + ap_2}$$

$$x_2 = \mathrm{I}\frac{a}{bp_1 + ap_2}$$

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Different Types of Preference

Preferences	Utility fn	Corner Soln?	Kink?	Tangent?
Cobb Douglas	$x_1^a x_2^b$ $a\ln x_1 + b\ln x_2$	Never	Never	Always
Perfect Sub	$a\,x_1 + b\,x_2$	Always	Never	Rarely
Perfect Comp	$\min(ax_1, bx_2)$	Never	Always	Never
Quasi Linear				
Concave				

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Quasi Linear Preferences

- Utility function: for $1 > a > 0$

$$u(x_1, x_2) = x_1^a + x_2$$

- Indifference curve:

$$x_2 = \mathrm{u} - x_1^a$$

- Note indifference curves are vertical translations of each other
- Also note that slope of indifference curve goes to infinity when x_1 goes to 0 but does not go to zero as x_2 goes to zero

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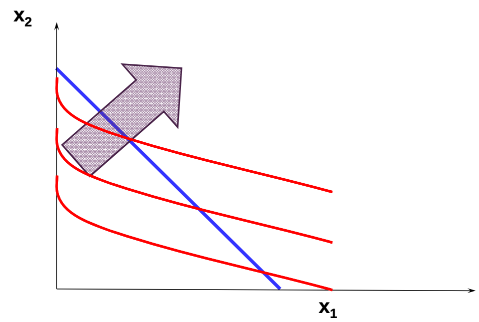
Quasi Linear Preferences



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Quasi Linear Preferences

- Can there be a Kink?
  - No, indifference curve is smooth
- Can there be a tangency solution?
  - Well, lets see….

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Quasi Linear Preferences

$$u(x_1, x_2) = u(x_1, x_2) = x_1^a + x_2$$

- Taking derivatives gives

$$MU_1 = \frac{\partial u}{\partial x_1} = ax_1^{a-1}$$

$$MU_2 = \frac{\partial u}{\partial x_2} = 1$$

- And so MRS is

$$\frac{\partial u}{\partial x_1} \Big/ \frac{\partial u}{\partial x_2} = ax_1^{a-1}$$

- Setting this equal to the price ratio gives

$$ax_1^{a-1} = \frac{p_1}{p_2} \text{ or } x_1 = \left(\frac{ap_2}{p_1}\right)^{\frac{1}{1-a}}$$

Quasi Linear Preferences

$$x_1 = \left(\frac{ap_2}{p_1}\right)^{\frac{1}{1-a}}$$

- Note that this isn't a function of income
- Implies

$$x_2 = \frac{I - \left(\frac{ap_2}{p_1}\right)^{\frac{1}{1-a}}}{p_2}$$

- Is this a solution?
- It depends on the parameters of the problem!

Quasi Linear Preferences

$$x_1 = \left(\frac{ap_2}{p_1}\right)^{\frac{1}{1-a}} \quad x_2 = \frac{I - \left(\frac{ap_2}{p_1}\right)^{\frac{1}{1-a}}}{p_2}$$

- Say $a = 1/2$, $p_1 = 1$, $p_2 = 2$ and I= 8,
- Then $x_1 = 1$ and $x_2 = \frac{7}{2}$

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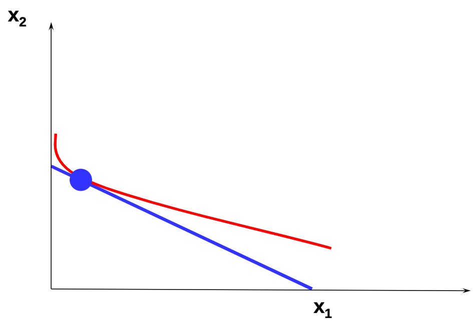
Quasi Linear Preferences



Quasi Linear Preferences

$$x_1 = \left(\frac{ap_2}{p_1}\right)^{\frac{1}{1-a}} \quad x_2 = \frac{I - \left(\frac{ap_2}{p_1}\right)^{\frac{1}{1-a}}}{p_2}$$

- Say $a = 1/2$, $p_1 = 1$, $p_2 = 5$ and I= 4,
- Then $x_1 = \frac{25}{4}$ and $x_2 = -\frac{9}{20}$
- !

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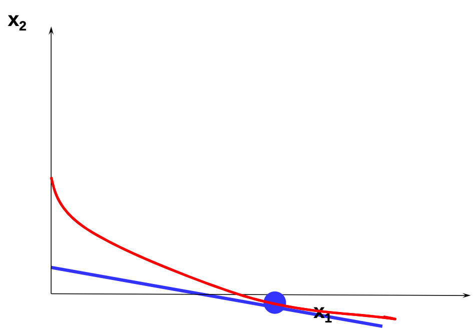
Quasi Linear Preferences



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Quasi Linear Preferences

- In the second case, tangency point occurs only when the amount of good 2 is negative
- No 'feasible' tangency point
- Solution will be at the corner

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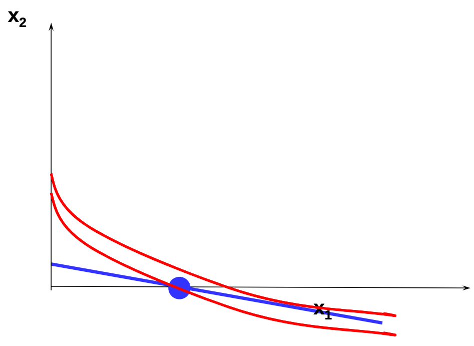
Quasi Linear Preferences



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Different Types of Preference

Preferences	Utility fn	Corner Soln?	Kink?	Tangent?
Cobb Douglas	$x_1^a x_2^b$ $a\ln x_1 + b\ln x_2$	Never	Never	Always
Perfect Sub	$a\,x_1 + b\,x_2$	Always	Never	Rarely
Perfect Comp	$\min(ax_1, bx_2)$	Never	Always	Never
Quasi Linear	$x_1^a + x_2$	Maybe	Never	Maybe
Concave				

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Concave Preferences

- Utility function: for $a, b > 1$
  $$u(x_1, x_2) = x_1^a + x_2^b$$
- Indifference curve:
  $$x_2 = (\mathrm{u} - x_1^a)^{\frac{1}{b}}$$

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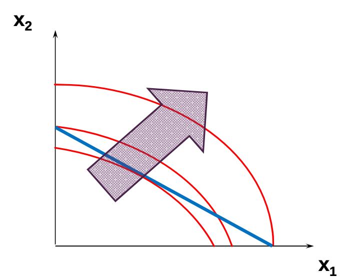
Concave Preferences



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Concave Preferences

- Can there be a Kink?
  - No, indifference curve is smooth
- Can there be a tangency solution?
  - Well, there **will** be a point of tangency

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Concave Preferences

$$u(x_1, x_2) = x_1^a + x_2^b$$

- Taking derivatives gives
  $$MU_1 = \frac{\partial u}{\partial x_1} = ax_1^{a-1}$$
  $$MU_2 = \frac{\partial u}{\partial x_2} = bx_2^{b-1}$$
- And so MRS is
  $$\frac{\partial u}{\partial x_1}\Big/\frac{\partial u}{\partial x_2} = \frac{ax_1^{a-1}}{bx_2^{b-1}}$$
- Setting this equal to the price ratio gives
  $$\frac{ax_1^{a-1}}{bx_2^{b-1}} = \frac{p_1}{p_2} \text{ or } x_2 = \left(\frac{ap_2}{bp_1}x_1^{a-1}\right)^{\frac{1}{b-1}}$$

Concave Preferences

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- But this will be a local **minimum**
- This can be seen from the indifference curves

Concave Preferences

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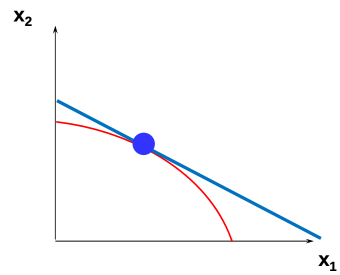

Concave Preferences

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- But this will be a local **minimum**
- This can be seen from the indifference curves
- And also from the second order conditions (see lecture 5)
- Solution will therefore be a corner solution

Concave Preferences

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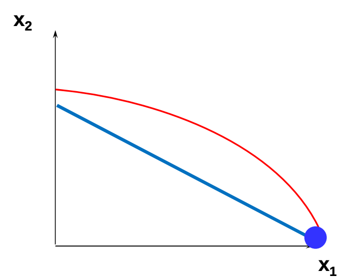

Different Types of Preference

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Preferences	Utility fn	Corner Soln?	Kink?	Tangent?
Cobb Douglas	$x_1^a x_2^b$ $a\ln x_1 + b\ln x_2$	Never	Never	Always
Perfect Sub	$a x_1 + b x_2$	Always	Never	Rarely
Perfect Comp	$\min(ax_1, bx_2)$	Never	Always	Never
Quasi Linear	$x_1^a + x_2$	Maybe	Never	Maybe
Concave	$x_1^a + x_2^b$	Yes	Never	Never

Income, Substitution and Slutsky

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The impact of price changes

- In lecture 6, we showed how a change in prices could be split into an income and substitution effect
- We then introduced the 'Slutsky equation', which showed how to make this decomposition formal
- Confusingly, the Slutsky equation uses a slightly different definition of the income and substitution effect
- Let's see if we can make this clearer
  - First, remind you what the income and substitution effect is
  - Then try to describe the Slutsky equation (without the maths)

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More about price changes

- What happens when a commodity's price decreases?

1. **Substitution effect**: the commodity is relatively cheaper, so consumers substitute it for now relatively more expensive other commodities.
2. **Income effect**: the consumer's budget of $y can purchase *more than before*, *as if the consumer's income rose*, with consequent income effects on quantities demanded.

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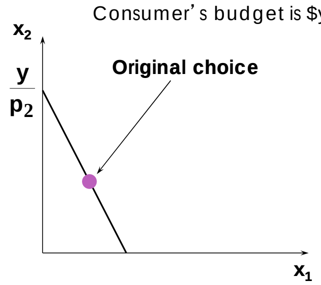
Effects of a Price Change

Consumer's budget is $y



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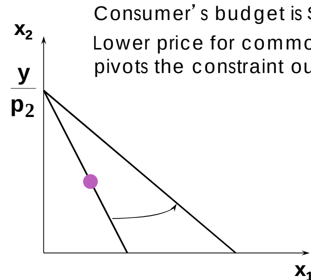
Effects of a Price Change

Consumer's budget is $y

Lower price for commodity 1 pivots the constraint outwards



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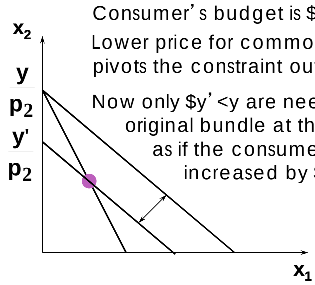
Effects of a Price Change

Consumer's budget is $y

Lower price for commodity 1 pivots the constraint outwards

Now only $y'<y are needed to buy the original bundle at the new prices, as if the consumer's income has increased by $y - $y'



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Pure Substitution Effect

- We will now calculate these two effects separately
- How?
- We 'break up' the movement of the budget line in two
- First, we move to a budget line with the **new slope** but so that the **original bundle was just affordable**
  - This is gives us a new budget set
  - We compute the bundle that would be chosen by the decision maker for this set

Pure Substitution Effect Only

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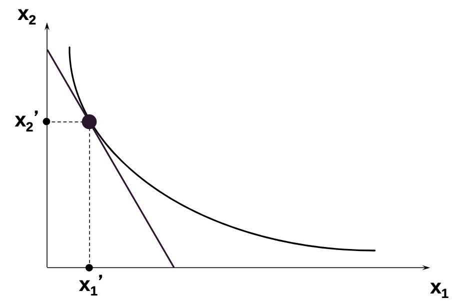

Pure Substitution Effect Only

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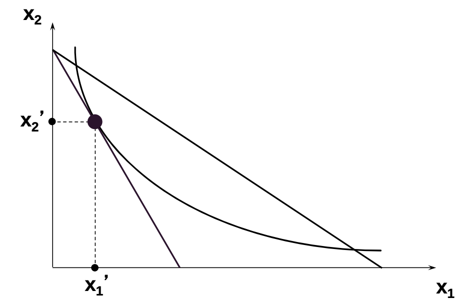
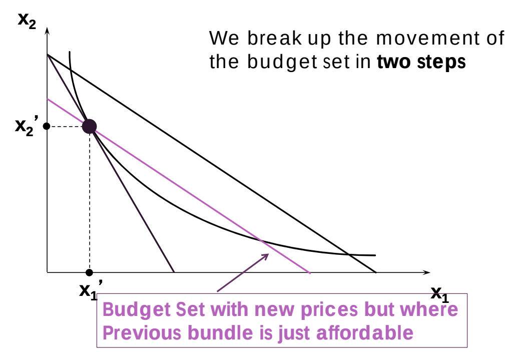


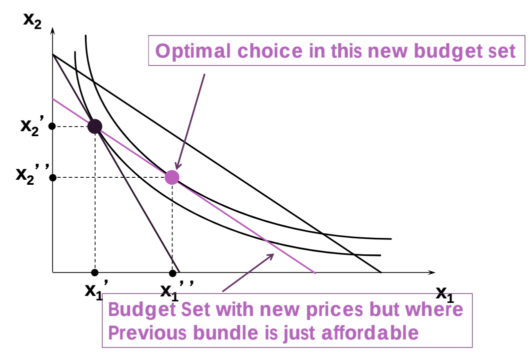
Pure Substitution Effect Only

75



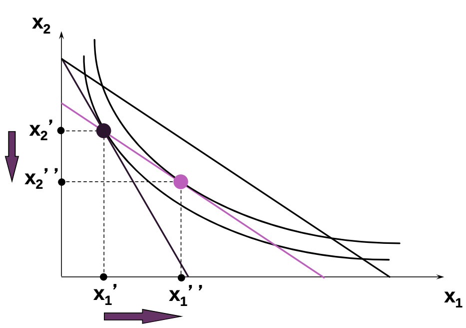
Pure Substitution Effect Only

76



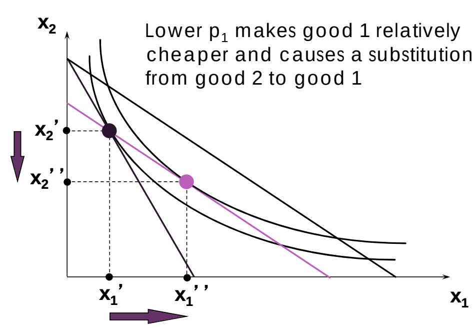
Pure Substitution Effect Only

77



Pure Substitution Effect Only

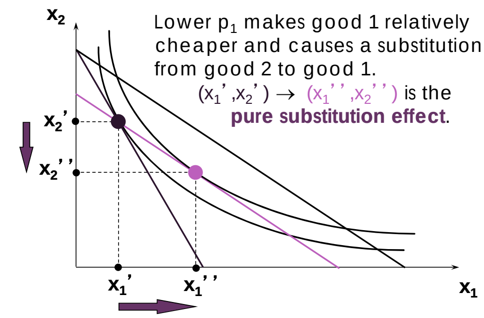
78



Pure Substitution Effect Only

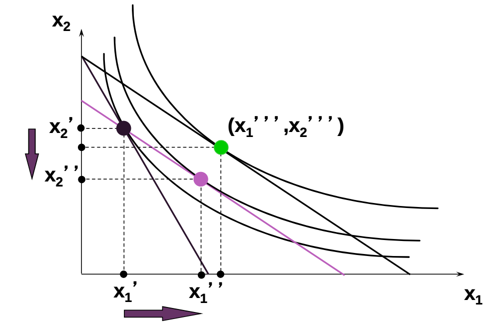
79

Lower p_1 makes good 1 relatively cheaper and causes a substitution from good 2 to good 1.
$(x_1', x_2') \to (x_1'', x_2'')$ is the **pure substitution effect**.



And Now The Income Effect

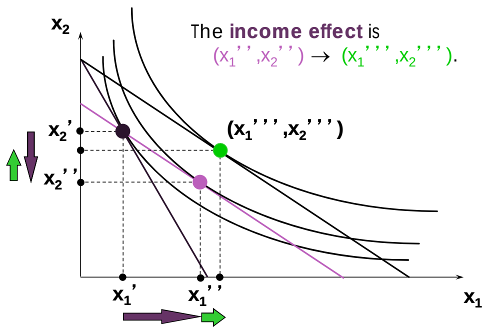
80



And Now The Income Effect

81

The **income effect** is $(x_1'', x_2'') \to (x_1''', x_2''')$.

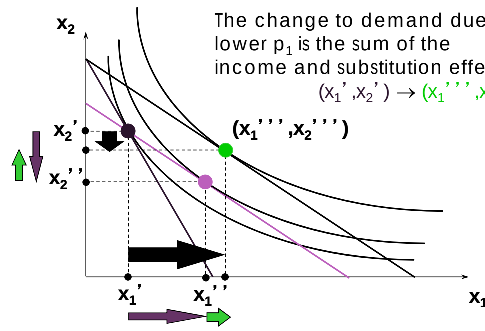


It is just like a change in income!

The Overall Change in Demand

82

The change to demand due to lower p_1 is the sum of the income and substitution effects, $(x_1', x_2') \to (x_1''', x_2''')$.



So:

83

- We can separate the change due to a price change into:
  - Substitution effect: effect of changing prices keeping income constant
    - i.e. keeping the old bundle just affordable at the new prices
  - Income effect: effect of the change in income at the new prices

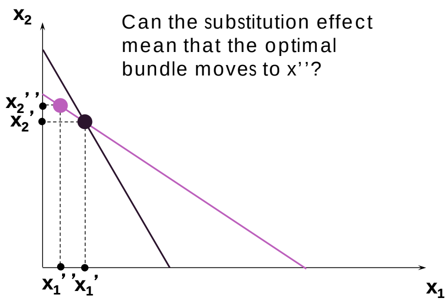
What are the signs of these effects?

84

- Are these effects always positive, negative ?
- First, how about the substitution effect?
  - It must be **always positive:** lower price lead to more demand in the substitution effect
  - Why? See graph

Pure Substitution Effect Only

85



Can the substitution effect mean that the optimal bundle moves to x''?

Pure Substitution Effect Only

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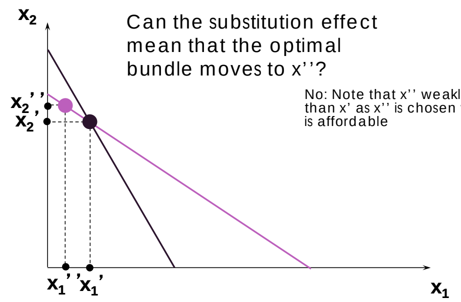
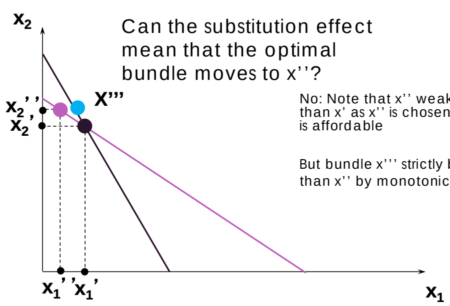


Can the substitution effect mean that the optimal bundle moves to x''?

No: Note that x'' weakly better than x' as x'' is chosen when x' is affordable

Pure Substitution Effect Only

87



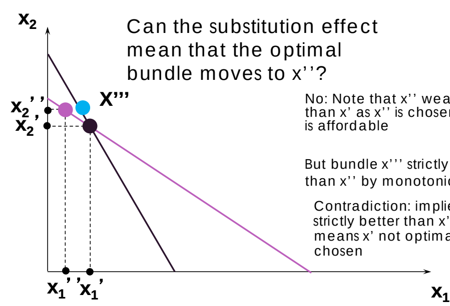
Can the substitution effect mean that the optimal bundle moves to x''?

No: Note that x'' weakly better than x' as x'' is chosen when x' is affordable

But bundle x''' strictly better than x'' by monotonicity

Pure Substitution Effect Only

88



Can the substitution effect mean that the optimal bundle moves to x''?

No: Note that x'' weakly better than x' as x'' is chosen when x' is affordable

But bundle x''' strictly better than x'' by monotonicity

Contradiction: implies x''' strictly better than x'. But that means x' not optimal when chosen

What are the signs of these effects?

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- Are these effects always positive?
- First, how about the substitution effect?
  - It must be **always positive:** lower price leads to more demand in the substitution effect
  - Why? See graph
- How about the income effect?
  - It's just like a change in income
  - Does demand go up or down with income?
  - If good is **inferior** -> negative
  - If good is **normal** -> positive

What are the signs of these effects?

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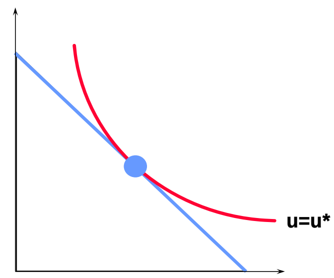
- So:
- **Substitution is always positive and income could be positive or negative (in the case of inferior goods)**
- Total effect is always the sum
- Means that: for normal goods, total effect is positive
- For inferior goods: if income effect is stronger than substitution effect, then total effect can be negative

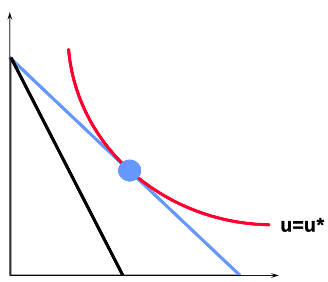
The Slutsky Equation

- Right, now for the Slutsky equation
- This is also going to break down the effect of a price change into an 'income' and 'substitution' effect
- But the definitions are going to be slightly different
  - Substitution effect: effect of changing prices keeping **utility** constant
    - i.e. keeping the old **indifference curve** at the new prices
  - Income effect: effect of the change in utility at the new prices

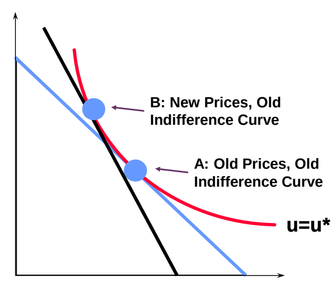
Increase in Price of Good 1



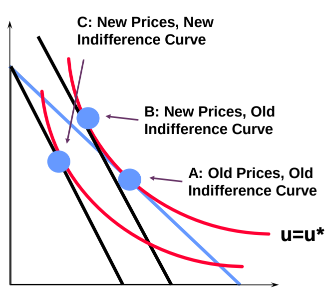
Increase in Price of Good 1



Increase in Price of Good 1



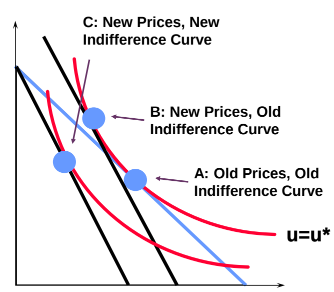
Increase in Price of Good 1



Increase in Price of Good 1

A to B: Substitution Effect

B to C: Income Effect



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The Slutsky Equation

- In order to define these effects mathematically, we need to define the consumer's dual problem

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Ordinary and Compensated Demand

- Here is the standard consumer problem

1. **CHOOSE** **a consumption bundle**
2. **IN ORDER TO MAXIMIZE** **preferences**
3. **SUBJECT TO** **the budget constraint**

- This gives rise to demand functions: amount of the good consumed given prices and income

$$x_i(p, y)$$

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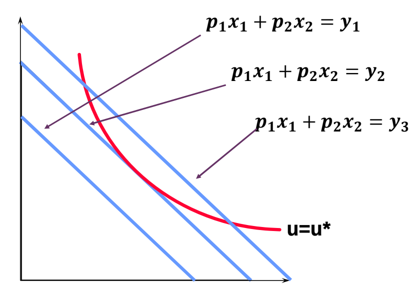
Ordinary and Compensated Demand

- Here is a related problem, sometimes called the 'dual' problem

1. **CHOOSE** **a consumption bundle**
2. **IN ORDER TO MINIMIZE** **expenditure**
3. **SUBJECT TO** **utility being equal to some u***

100

The Dual Problem



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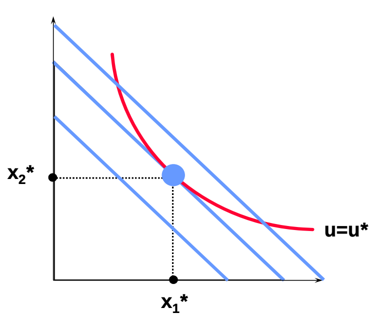
The Dual Problem



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Ordinary and Compensated Demand

- Here is a related problem, sometimes called the 'dual' problem

1. **CHOOSE** **a consumption bundle**
2. **IN ORDER TO MINIMIZE** **expenditure**
3. **SUBJECT TO** **utility being equal to some u***

- This gives rise to **compensated** demand functions: amount of the good consumed given prices and **utility**

$$x_i^h(p, u)$$

The Slutsky Equation

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- This is the Slutsky Equation

$$\frac{\partial x_1(p,y)}{\partial p_1}=\frac{\partial x_1^h(p,u)}{\partial p_1}-\frac{\partial x_1(p,y)}{\partial y}x_1^h(p,u)$$

- First term on the right hand side is the substitution effect: change in demand as price changes, keeping utility constant
  - Always negative

Increase in Price of Good 1

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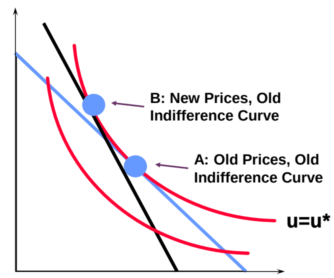
A to B: Substitution Effect



The Slutsky Equation

105

- This is the Slutsky Equation

$$\frac{\partial x_1(p,y)}{\partial p_1}=\frac{\partial x_1^h(p,u)}{\partial p_1}-\frac{\partial x_1(p,y)}{\partial y}x_1^h(p,u)$$

- First term is the substitution effect: change in demand as price changes, keeping utility constant
  - Always negative
- Second term is the income effect: change in demand due to effecting change in income
  - Can be positive or negative

Increase in Price of Good 1

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B to C: Income Effect