THE IMPLICIT FUNCTION THEOREM

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Theorem 1 *(IFT) Let* $(x^*, y^*) \in \mathbb{R}^2$ *and let* $G : \mathbb{R}^2 \to \mathbb{R}$ *be a* C^1 *function in a neighborhood of* (x^*, y^*) *with* $G(x^*, y^*) = 0$. *If* $\left.\frac{\partial G}{\partial y}\right|_{(x^*,y^*)} \neq 0$, *then there exists a* C^1 *function* f *in a neighborhood* I *of* x^* *such that*

$$\begin{aligned} (a) \qquad G(x, f(x)) &= 0 \text{ for all } x \in I \\ (b) \qquad f(x^*) &= y^*, \text{ and} \\ (c) \qquad f'(x^*) &= -\frac{\left.\frac{\partial G}{\partial x}\right|_{(x^*,y^*)}}{\left.\frac{\partial G}{\partial y}\right|_{(x^*,y^*)}} \end{aligned}$$

Proof. Assume WLOG that $\left.\frac{\partial G}{\partial y}\right|_{(x^*,y^*)} > 0$

Then G is strictly increasing in y in a neighborhood of (x^*, y^*). Let $\varepsilon > 0$ we have that $G(x^*, y^* - \varepsilon) < 0$ and $G(x^*, y^* + \varepsilon) > 0$ [Why?]. We also must have that $G(x, y^* - \varepsilon) < 0$ and $G(x, y^* + \varepsilon) > 0$ for all x in a neighborhood I of x^*.[Why?]. If necessary we can restrict I so that $\frac{\partial G}{\partial y} > 0$ on $I \times (y^* - \varepsilon, y^* + \varepsilon) \neq 0$ [Why can we do this?].

Then $\forall x \in I, \exists! y_x \in (y^* - \varepsilon, y^* + \varepsilon)$ such that $G(x, y_x) = 0$, since G is continuous and strictly increasing in y.

Setting $f(x) = y_x$ for all $x \in I$ we have the Implicit function defined.
(a) and (b) are satisfied by construction.
I'm omitting the proof that $f \in C^1(I)$, taking this for granted (c) follows by the chain rule. ■

Theorem 2 (IFT $\mathbb{R}^k$ version) *Let* $z^* = (x_1^*, ..., x_k^*, y^*) \in \mathbb{R}^{k+1}$ *and let* $G : \mathbb{R}^{k+1} \to \mathbb{R}$ *be a* C^1 *function in a neighborhood of* z^* *with* $G(z^*) = 0$. *If* $\left.\frac{\partial G}{\partial y}\right|_{z^*} \neq 0$, *then there exists a* C^1 *function* f *in a neighborhood* I *of* $(x_1^*, ..., x_k^*)$ *such that*

$$\begin{aligned} (a) \qquad G(x, f(x)) &= 0 \text{ for all } x \in I \\ (b) \qquad f(x_1^*, ..., x_k^*) &= y^*, \text{ and} \\ (c) \quad \frac{\partial f}{\partial x_i}(x_1^*, ..., x_k^*) &= -\frac{\left.\frac{\partial G}{\partial x_i}\right|_{z^*}}{\left.\frac{\partial G}{\partial y}\right|_{z^*}} \end{aligned}$$

Example

We often work with general functions rather than a particular functional. For instance, let's consider the intertemporal utility maximization problem for an individual that onlly derived utility from consumption, whose stream of income is given by (w_1, w_2) and can save at an exogenous interest rate r. Let assume that we are interested in studying the dynamics of savings, in particular how savings respond to changes in the interest rate. The FOC of this individual is given by $G = -u'(w_1 - s) + (1+r)u'(w_2 + (1+r)s) = 0$. We are interested in knowing ds/dr that we can determine using the IFT:

$$\frac{ds}{dr} = -\frac{\frac{\partial G}{\partial r}}{\frac{\partial G}{\partial s}} = -\frac{u'(w_2 + (1+r)s) + (1+r)\,u''\,(w_2 + (1+r)s)\,s}{u''(w_1 - s) + (1+r)^2 u''(w_2 + (1+r)s)}$$

Applications

- - Comparative statics in general
  - Rates of substitution: MRS or MRTS