Linear Algebra Solutions[1]

Math Camp 2012

Do the following:

1. Let $A = \begin{pmatrix} 2 & 0 \\ 3 & 8 \end{pmatrix}$ and $B = \begin{pmatrix} 7 & 2 \\ 6 & 3 \end{pmatrix}$. Find $A - B$, $A + B$, AB, and BA.

   $A - B = \begin{pmatrix} -5 & -2 \\ -3 & 5 \end{pmatrix}$

   $A + B = \begin{pmatrix} 9 & 2 \\ 9 & 11 \end{pmatrix}$

   $AB = \begin{pmatrix} 14 & 4 \\ 69 & 30 \end{pmatrix}$

   $BA = \begin{pmatrix} 20 & 16 \\ 21 & 24 \end{pmatrix}$

2. Let $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $u = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$. Find $u \cdot v$, $u'v$ and $v'u$.

   $uv = 13$

   $u'v = 13$

   $v'u = 13$

3. Prove that the multiplication of any matrix with its transpose yields a symmetric matrix

   The i, j^{th} element of AA^T, where A is $m \times n = \sum_{k=1}^{n} = \sum_{k=1}^{n}(a_{ik}a_{jk}) = \sum_{k=1}^{n}(a_{jk}a_{ik})$

   The j, i^{th} element of AA^T is $\sum_{k=1}^{n} = \sum_{k=1}^{n}(a_{jk}a_{ki}^{T}) = \sum_{k=1}^{n}(a_{jk}a_{ik})$

   Therefore $(AA^T)_{i,j} = (AA^T)_{j,i}$, therefore A is symmetric. QED.

4. Prove that A only has an inverse if it is a square matrix.

   Suppose, by contradiction that A is $m \times n$ and there exists A^{-1}, where $m \neq n$. If B is an inverse of A, then $AB = I_{m \times m}$ therefore B has to have m rows. But also it would be the case that $BA = I_{n \times n}$ therefore B has to have n columns, contradiction.

5. Prove the first four properties of transpose matrices above.

   The definition of the transpose of a matrix is given by A' is a matrix such that for each element of A', $a'_{i,j} = a_{j,i}$

   (a) $(A')' = A$.

   Let $B = A'$, we want to prove that $B' = A$. By the definition of the transpose we have that $b_{i,j} = a_{j,i}$. Using the definition of the transpose again we have that B', $b'_{i,j} = b_{j,i} = a_{i,j}$. QED.

[1] If you find any typo please email me: Maria_Jose_Boccardi@Brown.edu

(b) $(A+B)' = A' + B'$.

The element i,j^{th} of the matrix $A+B$ is given by $a_{i,j}+b_{i,j}$, therefore the element i,j^{th} of $(A+B)'$ is given by

$$a'_{i,j} + b'_{i,j} = a_{j,i} + b_{j,i}$$

where $a_{j,i}$ and $b_{j,i}$ are the elements i,j^{th} of the matrices A' and N' respectively.

(c) $(\alpha A)' = \alpha A'$.

Let $B = \alpha A$, by the definition of the transpose we know that $(\alpha A)' = (b_{i,j})' = b_{j,i} = \alpha a_{j,i} = \alpha(a'_{i,j})$

(d) $(AB)' = B'A'$.

$$(AB)'_{i,j} = (AB)_{j,i} = \sum_{k=1}^{n} a_{jk}b_{ki} = \sum_{k=1}^{n} a'_{kj}b'_{ik} = \sum_{k=1}^{n} b'_{ik}a'_{kj} = (B'A')_{ij}$$

6. In econometrics, we deal with a matrix called the projections matrix: $A = I - X\left(X'X\right)^{-1}X'$. Must A be square? Must $X'X$ be square? Must X be square?

A has to be square, $X'X$ has to be square, X does not have to be square.

7. Show that the projection matrix in 6 is idempotent.

We need to show that $AA = A$ and $A^T = A$.

$$\begin{aligned} AA &= (I - X\left(X'X\right)^{-1}X')(I - X\left(X'X\right)^{-1}X') \\ &= I - X\left(X'X\right)^{-1}X' - X\left(X'X\right)^{-1}X' + X\left(X'X\right)^{-1}X'X\left(X'X\right)^{-1}X' \\ &= I - X\left(X'X\right)^{-1}X' = A \end{aligned}$$

$$\begin{aligned} A' &= \left(I - X\left(X'X\right)^{-1}X'\right)' \\ &= I' - \left(X\left(X'X\right)^{-1}X'\right)' \\ &= I - X\left(X\left(X'X\right)^{-1}\right)' \\ &= I - X\left(\left(X'X\right)^{-1}\right)'X' \\ &= I - X\left(X'X\right)^{-1}X' \end{aligned}$$

8. Calculate the determinants of the following matrices:

(a) $\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$=1

(b) $\begin{pmatrix} -3 & 4 \\ 4 & -5 \end{pmatrix}$=-1

(c) $\begin{pmatrix} -3 & 4 \\ 4 & -6 \end{pmatrix}$=2

(d) $\begin{pmatrix} 2 & 4 \\ 4 & 8 \end{pmatrix}$=0

(e) $\begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 6 \end{pmatrix}$=-25

(f) $\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$=0

(g) $\begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 2 & 0 & 5 \\ 3 & 0 & 4 & 0 \\ 0 & 5 & 0 & 6 \end{pmatrix}$=65

9. Evaluate the following determinants:

   (a) $\begin{vmatrix} 1 & 1 & 4 \\ 8 & 11 & -2 \\ 0 & 4 & 7 \end{vmatrix}$=157

   (b) $\begin{vmatrix} 1 & 2 & 0 & 9 \\ 2 & 3 & 4 & 6 \\ 1 & 6 & 0 & -1 \\ 0 & -5 & 0 & -8 \end{vmatrix}$=328

10. Find the inverse of the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

    $$\begin{pmatrix} 1 & 0 & 0 & | & 1 & 0 & 0 \\ 0 & 0 & 1 & | & 0 & 1 & 0 \\ 0 & 1 & 0 & | & 0 & 0 & 1 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1 & 0 & 0 & | & 1 & 0 & 0 \\ 0 & 1 & 0 & | & 0 & 0 & 1 \\ 0 & 0 & 1 & | & 0 & 1 & 0 \end{pmatrix}$$

    Therefore $A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

11. Prove that for a 3×3 matrix, one may find the determinant by a cofactor expansion along any row or column in the matrix.

    Just do the calculations and prove that all the formulas are the same.

12. Determine the ranks of the matrices below. How many linearly independent rows are in each? Which have inverses?

    (a) $\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$

    Rank =2 and it is invertible since the determinant is different from zero.

    (b) $\begin{pmatrix} -3 & 4 \\ 4 & -5 \end{pmatrix}$

    Rank =2 and it is invertible since the determinant is different from zero.

    (c) $\begin{pmatrix} -3 & 4 \\ 4 & -6 \end{pmatrix}$

    Rank =2 and it is invertible since the determinant is different from zero.

(d) $\begin{pmatrix} 2 & 4 \\ 4 & 8 \end{pmatrix}$

Rank =1 and it is not invertible since the determinant is zero.

(e) $\begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 6 \end{pmatrix}$

Rank =3 and it is invertible since the determinant is different from zero.

(f) $\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$

Rank =2 and it is not invertible since the determinant is zero. (Column 2 and 3 are LD)

(g) $\begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 2 & 0 & 5 \\ 3 & 0 & 4 & 0 \\ 0 & 5 & 0 & 6 \end{pmatrix}$

Rank =4 and it is invertible since the determinant is different from zero.

Quadratic forms

1. Express the quadratic form as a matrix product involving a symmetric coefficient matrix.

   (a) $Q = 8x_1x_2 - x_1^2 - 31x_2^2$

   $\begin{pmatrix} -1 & 4 \\ 4 & -31 \end{pmatrix}$

   (b) $Q = 3x_1^2 - 2x_1x_2 + 4x_1x_3 + 5x_2^2 + 4x_3^2 - 2x_2x_3$

   $\begin{pmatrix} 3 & -1 & 2 \\ -1 & 5 & -1 \\ 2 & -1 & 4 \end{pmatrix}$

2. List all the principal minors of the 4×4 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

First order principal minors: a_{11}, a_{22}, a_{33} and a_{44}.

Second order principal minors:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$\begin{pmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{pmatrix}$$

$$\begin{pmatrix} a_{22} & a_{24} \\ a_{42} & a_{44} \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{14} \\ a_{41} & a_{44} \end{pmatrix}$$

$$\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{pmatrix}$$

The third order principal minors are:

$$\begin{pmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{13} & a_{14} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{41} & a_{42} & a_{44} \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

The fourth order principal minor is A

3. Prove that:

   (a) Every diagonal matrix whose diagonal elements are all positive is positive definite.

   If every element in the diagonal in a diagonal matrix are positive then we have that

   $$x'Ax = \sum_{i=1}^{n} a_{ii}x_i^2 > 0$$

   for all $x \neq 0$

   (b) Every diagonal matrix whose diagonal elements are all negative is negative definite.

   Same argument as before

   (c) Every diagonal matrix whose diagonal elements are all positive or zero is positive semidefinite.

   Same argument as before

   (d) Every diagonal matrix whose diagonal elements are all negative or zero is negative semidefinite.

   Same argument as before.

   (e) All other diagonal matrices are indefinite.

4. Determine the definiteness of the following matrices:

(a) $\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$

Positive definite

(b) $\begin{pmatrix} -3 & 4 \\ 4 & -5 \end{pmatrix}$

Indefinite

(c) $\begin{pmatrix} -3 & 4 \\ 4 & -6 \end{pmatrix}$

negative definite

(d) $\begin{pmatrix} 2 & 4 \\ 4 & 8 \end{pmatrix}$

Positive semidefinite

(e) $\begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 6 \end{pmatrix}$

indefinite

(f) $\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$

negative semidefinite

(g) $\begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 2 & 0 & 5 \\ 3 & 0 & 4 & 0 \\ 0 & 5 & 0 & 6 \end{pmatrix}$

indefinite

1. Find the eigenvalues and the corresponding eigenvectors for the following matrices:

(a) $\begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$

$\lambda_1 = -1$ and the associated eigenvector is $v_1 = (0, 1)$

$\lambda_2 = 3$ and the associated eigenvector is $v_2 = (3, 2)$

(b) $\begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix}$

$\lambda_1 = 4$ and the associated eigenvector is $v_1 = (\frac{3}{2}, 1)$

(c) $\begin{pmatrix} 0 & 3 \\ 4 & 0 \end{pmatrix}$

$\lambda_1 = 2\sqrt{3}$ and the associated eigenvector is $v_1 = (\frac{1}{2}\sqrt{3}, 1)$

$\lambda_2 = -2\sqrt{3}$ and the associated eigenvector is $v_2 = (-\frac{1}{2}\sqrt{3}, 1)$

(d) $\begin{pmatrix} -2 & -7 \\ 1 & 2 \end{pmatrix}$

It has not real eigenvalues, the complex are: $\lambda_{1,2} = \pm\sqrt{3}i$

(e) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\lambda_1 = 0$ and the associated eigenvectors are $v_1 = (1,0)$ and $v_2 = (0,1)$

(f) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\lambda_1 = 1$ and the associated eigenvectors are $v_1 = (1,0)$ and $v_2 = (0,1)$

2. Determine whether the following matrices are diagonalizable:

(a) $\begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$

NO

(b) $\begin{pmatrix} 2 & -3 \\ 1 & -1 \end{pmatrix}$

NO (if we are considering matrices with only real entries)

(c) $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}$

NO

(d) $\begin{pmatrix} -1 & 0 & 1 \\ -1 & 3 & 0 \\ -4 & -13 & -1 \end{pmatrix}$

NO (if we are considering matrices with only real entries)

(e) $\begin{pmatrix} 2 & -1 & 0 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 3 \end{pmatrix}$

NO

3. Diagonalize the following matrices, if possible:

(a) $\begin{pmatrix} -14 & 12 \\ -20 & 17 \end{pmatrix}$

$P = \begin{pmatrix} \frac{4}{5} & \frac{3}{4} \\ 1 & 1 \end{pmatrix}$

$P^{-1} = \begin{pmatrix} 20 & -15 \\ -20 & 16 \end{pmatrix}$

$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 0 \\ 6 & -1 \end{pmatrix}$

$P = \begin{pmatrix} 0 & \frac{1}{3} \\ 1 & 1 \end{pmatrix}$

$P^{-1} = \begin{pmatrix} -3 & 1 \\ 3 & 0 \end{pmatrix}$

$\Lambda = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

$$P = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} 0 & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$\Lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

(d) $\begin{pmatrix} 2 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

$$P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}$$

$$P^{-1} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$\Lambda = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

4. (midterm exam) You have the following transition probability matrix of a discrete state Markov chain:

$$\begin{pmatrix} \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

What is the probability that the system is in state 2 at time $n+2$ given that it was in state 3 at time n?

We want to find $Prob(S_{n+2} = 2|S_n = 3)$ that is

$$\begin{aligned} Prob(S_{n+2} = 2|S_n = 3) &= Prob(S_{n+2} = 2|S_{n+1} = 1)Prob(S_{n+1} = 1|S_n = 3) \\ &+ Prob(S_{n+2} = 2|S_{n+1} = 2)Prob(S_{n+1} = 2|S_n = 3) \\ &+ Prob(S_{n+2} = 2|S_{n+1} = 3)Prob(S_{n+1} = 3|S_n = 3) \\ &+ Prob(S_{n+2} = 2|S_{n+1} = 4)Prob(S_{n+1} = 4|S_n = 3) \\ &= \frac{1}{4}\frac{1}{2} + 1 \times 0 + 0 \times \frac{1}{2} + \frac{1}{4} \times 0 = \frac{1}{8} \end{aligned}$$

5. (Homework problem) Suppose that the weather can be sunny or cloudy and the weather conditions on succesive mornings form a Markov chain with stationary transition probabilities. Suppose that the transition matrix is as follows

$$\begin{pmatrix} .7 & .3 \\ .6 & .4 \end{pmatrix}$$

where sunny is state 1 and cloudy is state 2. If it is cloudy on a given day, what is the probability that it will also be cloudy the next day?

$$Prob(S_{t+1} = 2|S_t = 2) = .4$$

6. (Homework problem) Suppose that three boys 1, 2, and 3 are throwing the ball to one another. Whenever 1 has the ball, he throws it to 2 with a probability of 0.2. Whenever 2 has the ball, he will throw it to 1 with probability 0.6. Whenever 3 has the ball, he is equally likely to throw it to 1 or 2.

   (a) Construct the transition probability matrix

   $$P = \begin{pmatrix} 0 & 0.2 & 0.8 \\ 0.6 & 0 & 0.4 \\ 0.5 & 0.5 & 0 \end{pmatrix}$$

   (b) If each of the boys is equally likely to have the ball at a certain time n, which boy is most likely to have the ball at time $n+2$?

   $x_n = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ therefore we have that

   $$x_{n+2} = xPP = \begin{pmatrix} \frac{11}{30} & \frac{7}{30} & \frac{2}{5} \end{pmatrix} P = \begin{pmatrix} \frac{17}{50} & \frac{41}{150} & \frac{29}{75} \end{pmatrix}$$

   therefore Boy 3 is most likely to have the ball at time n+2