Mathematics For Economists

Mark Dean

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Question 1 (10 Points) Let f be a continuously differentiable function on an interval I in $\mathbb{R}$. Show that f is concave if and only if

$$f(y) - f(x) \leq f'(x)(y - x)$$

Show that this implies that , if f is a continuously differentiable function on a convex subset U of $\mathbb{R}^n$, then f is concave on U if and only if, for all $x, y \in U$

$$f(y) - f(x) \leq \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f(x)(y_i - x_i)$$

Question 2 (20 Points) Let $f : R^n \to R$ be a concave and continuously differentiable function, and $g_j : R^n \to R$, $j = 1, ..., m$, be convex and continuously differentiable functions. Consider the optimization problem (P):

$$\begin{aligned} & \max\left(f\left(x\right)\right) \\ & s.t. \\ g\left(x\right) & \leq 0 \end{aligned}$$

Let $X = R^n$, $U = R^n_-$ and $l : X \times U \to R$ be the (Langrangian) function

$$l\left(x, \mu\right) = f\left(x\right) + \Sigma_j \mu_j g_j\left(x\right)$$

A saddle point for l is a vector $(x^*, \mu^*) \in X \times U$ that satisfies

$$l\left(x, \mu^*\right) \leq l\left(x^*, \mu^*\right) \leq l\left(x^*, \mu\right)$$

for all $(x, \mu) \in X \times U$.

1. Show that if (x^*, μ^*) is a saddle point for l then x^* is an optimal solution for (P).

2. Conversely, assume that (P) satisfies the Slater constraint qualification condition (SCQ) ; there exhausts $\overline{x}$ such that $g(\overline{x}) < 0$. Show that if x^* solves (P) then there exists a μ^* such that (x^*, μ^*) is a saddle point for l.(hint, use the answer to question 1

3. Show directly that (x^*, μ^*) is a saddle point for l if and only if it satisfies the KKT conditions.

Question 3 (40 Points) Let $u : \mathbb{R}^n_+ \to \mathbb{R}$ be a continuous utility function on an n-dimentional commodity space, and $B : \mathbb{R}_{++} \Rightarrow \mathbb{R}^n$ be the budget constraint defined by $B(p) = \{x \in \mathbb{R}^n | px \leq I\}$ for some I

1. Consider the problem of a consumer that has to spend at least k_i on each good i. Write down an expression for the choice set of the consumer as a function of p. Call this set $B^*(p)$. Define a set $P \in \mathbb{R}^n_{++}$ such that, for $p \in P$, $B^*(p)$ is non-empty.

2. Let $v : P \to \mathbb{R}$ be defined as $v(p) = \max_{x \in B^*(p)} u(p)$ and $d : P \Rightarrow \mathbb{R}$ be defined as $d(p) = \arg\max_{x \in B^*(p)} u(p)$. Can we guarantee that $v(p)$ and $d(p)$ are well defined? What about if the consumer had to spend more that k_i on each good? In either case prove that the v and d are well defined, or give an example where they are not

3. Assume once again that the consumer has to spend at least k_i on each good. Can we conclude that $v(p)$ is continuous? what about $d(p)$? Again, either prove that they are, or give an example in which they are not

4. Now say that the consumer only has to spend k_i on good i if the price of i is below some level $\bar{p}_i$ (i.e. if the price of good i goes above $\bar{p}_i$,then there is no restriction on how much has to be spent on i). Can we guarantee that v is continuous?

5. Now forget the additional condition in section 4 (i.e. return to the conditions in section 3). Let $n = 2$, and assume that $u(x_1, x_2) = x_1^\alpha x_2^\beta$. Can we conclude that the KKT conditions are necessary and sufficient for a local maximum? Will any local maximum also be a global maximum? Fix p such that $p_1 = p_2 = 1$ and $I = 2$. Find conditions on α, β k_1 and k_2 such that $\arg\max_{x \in B^*(p)} u(p) = \arg\max_{x \in B(p)} u(p)$

Question 4 (30 Points) An affine hull is defined as follows: For a set B in a linear space V, the affine hull of B is the smallest affine manifold that contains B.

1. Show that this means that

$$aff(B) = \cap \{M \subset V | M \text{ is an affine manifold and } B \subset M\}$$

2. Show that this also means that

$$aff(B) = \left\{ \sum_{i=1}^{k} \lambda_i x^i | k \in \mathbb{N},\ x_1, .., x_n \in B \text{ and } \lambda \in p_k \right\}$$

   Hint - you can use the result that a set S is an affine manifold if and only if $\lambda x + (1-\lambda)y \in S$ for all $x, y \in S$, $\lambda \in \mathbb{R}$

3. Calculate the affine manifolds of the following sets in $\mathbb{R}^3$

   (a) $S = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}$

   (b) $S = \left\{ x | x_1 = 1 \text{ and } (x_2 - 1)^2 + (x_2 - 1)^2 \leq 1 \right\}$

   (c) $S = \left\{ x | x_1^2 + x_2^2 + x_3^2 \leq 2 \right\}$

4. We define the concept of the relative interior of a set as follows: Let S a subset of a linear space V. A vector $x \in S$ is called a relative interior point of S if, for any $y \in aff(S)$ there exists a $\alpha_y > 0$ such that

$$(1-\alpha)x + \alpha y \in S \text{ for all } 0 \leq \alpha \leq \alpha_y$$

   The set of all relative interior points of S is called the relative interior of S. We denote this $ri(S)$

   (a) Let S be a set in $\mathbb{R}^n$. Show that, if $x \in int(S)$ then $x \in ri(S)$

   (b) See if you can draw a picture of a set S that contains a point that is in $x \in ri(S)$ but is not in $int(S)$

   (c) Calculate the relative interiors and interiors for the sets in part

5. Prove the following: Let C be a convex set in some metric space M. Let $x_1 \in ri(C)$ and $x_2 \in cl(C)$. Then $[x_1, x_2) \subset ri(c)$