Mathematics For Economists

Mark Dean

Final

December 12th 2013

REMAIN CALM

NOTE: Theorems in class notes can be taken as given. Prove all other statements

PLEASE WRITE QUESTIONS 1,2 AND 3 IN ONE BOOK AND 4 AND 5 IN ANOTHER

Question 1 (5 Points) Define $f : \mathbb{Q} \to \mathbb{R}$ as $f(q) = q$. Does the problem $\max_{q \in \mathbb{Q}} f(q)$ subject to $0 \leq q \leq \sqrt{2}$ have a solution? Either find the solution or prove that it does not have one. If not, which of the assumptions of Weierstrass theorem do not hold?

Question 2 Consider the following preference relation on $\mathbb{R}^2$:

$$\begin{aligned} (x_1, x_2) &\succeq (y_1, y_2) \\ \text{if } \min(\{x_1, x_2\}) &\geq \min(\{y_1, y_2\}) \end{aligned}$$

1. (5pts) Is this preference relation complete, continuous and reflexive (to do so, you may need to prove that the minimum of continuous functions is continuous)?
2. (15pts) Consider a firm who uses two inputs $k \in \mathbb{R}_+$ and $l \in \mathbb{R}_+$. They have a fixed budget B to spend on these inputs. Output $y(k, l)$ is given by $y(k, l) = \min(\{k, l\})$. Let $p_k, p_l \in \mathbb{R}_{++}$ be the price of each input,

$$Y(B, p_k, p_l) = \max\{y(k, l) \in \mathbb{R} | y = \min(\{k, l\}), p_k k + p_l l \leq B\}$$

and

$$D(B, p_k, p_l) = \arg\max\{y(k, l) \in \mathbb{R} | y = \min(\{k, l\}), p_k k + p_l l \leq B\}$$

Are Y and D always well defined in $\mathbb{R}$ and $\mathbb{R}^2$ respectively? Is D a function or a correspondence? are Y and D continuous (either prove that they are or find a counterexample)? How do your answers change if $p_k \in \mathbb{R}_+$? or if p_k,$p_l \in \mathbb{R}_+$?

3. (10pts) Now imagine that the government charges a tax on capital so that the price that the consumer faces is a function $T(p_k)$ of the price p_k. Reformulate $Y(B, p_k, p_l)$ and $D(B, p_k, p_l)$ appropriately, and answer the above questions for the following two cases (assume that p_k,$p_l \in \mathbb{R}_{++}$)

   (a) $T(p_k) = \alpha p_k$, $\alpha > 0$

   (b) $T(p_k) = p_k$ if $p_k \leq \bar{p}$, $T(p_k) = T + p_k$ for $T > 0$, $\bar{p} \in \mathbb{R}_{++}$

Question 3 Has two parts

1. (5 pts) Let $\mathcal{C}([0,1])$ (with the standard addition and scalar multiplication defined in class) denote the space of continuous real valued functions on the interval $[0,1]$, and let $\{x_0, x_1, ..., x_n\} \subset [0,1]$. For arbitrary $\{f_0, ..., f_l\} \subset \mathcal{C}[0,1]$, define $y^k \in \mathbb{R}^{n+1}$ by $y^k = (f_k(x_0), ..., f_k(x_n))$. Argue that $\{y^0, ..., y^l\}$ linearly independent implies $\{f_0, ..., f_l\}$ linearly independent.

2. (5 pts) Show that, for $V = \mathbf{P}^n([-1,1])$ (i..e the set of polynomials of degree n defined on $[-1,1]$) the following is an inner product

$$< p, q >= \int_{-1}^{1} p(x)q(x)dx$$

Question 4 A bird is trying to consider the best place to hover and look for prey coming in and out of a particular burrow. It's nest is at location $(0,0,1)$ (where the dimensions are latitude, longitude and height). The bird cannot stray more that a distance δ from its nest. Because of predators, it must hover at exactly height h above the ground. The burrow is located at $(1,1,0)$, and the probability of catching prey if the bird hovers at location x is given by $p(x) = \frac{1}{1+d(x,(1,1,0))}$ where $d(.,.)$ is the Euclidian distance

1. (10pts) Formulate this problem as an optimization problem. Provide conditions under which the KKT first order conditions are both necessary and sufficient to find an optimum

2. (5pts) Provide conditions on δ and h such that both constraints are binding

3. (10pts) Solve the problem for $\delta = 1$ an $h = 0.8$
4. (10 pts) Calculate the derivative of the probability of catching prey as a function of δ and h in the above problem, assuming that the bird accurately solves its optimization problem.

Question 5 Here we will derive a representation for a linear subspace.of a Euclidian space.

1. (6pts) Let Y be a k dimensional subspace of $\mathbb{R}^n$. Show that dim $Y^{\perp} = n - k$ (hint, use the orthogonal projection theorem)
2. (6pts) Show that $(Y^{\perp})^{\perp} = Y$
3. (6pts) Let $\{x^1 ... x^{n-k}\}$ be a basis for $Y^{\perp}$. Use (2) above to show

$$Y = \left\{ y \in \mathbb{R}^n | x^i . y = 0, \; i = 1, ..., n - k \right\}$$

4. (2pts) Conclude that there exists a matrix A such that

$$Y = [y \in \mathbb{R}^n | Ay = 0]$$