Mathematics For Economists

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Question 1 (15 Points) Let f be a continuously differentiable function on an interval I in $\mathbb{R}$. Show that f is concave if and only if

$$f(y) - f(x) \leq f'(x)(y - x)$$

Use this result to show that if f is a continuously differentiable function on a convex subset U of $\mathbb{R}^n$, then f is concave on U if and only if, for all $x, y \in U$ (for this second part, you may want to prove that f is concave if the function $g_{x,y}(t) = f(tx + (1-t)y)$ is concave for every $x, y \in U$)

$$f(y) - f(x) \leq \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f(x)(y_i - x_i)$$

Answer: If f is concave, then

$$\begin{aligned} f(ty + (1-t)x) &\geq tf(y) + (1-t)f(x) \\ &\Rightarrow f(x + t(y-x)) \geq t(f(y) - f(x)) + f(x) \\ &\Rightarrow \frac{f(x + t(y-x)) - f(x)}{t} \geq f(y) - f(x) \\ &\Rightarrow \frac{f(x + t(y-x)) - f(x)}{t(y-x)}(y-x) \geq f(y) - f(x) \end{aligned}$$

Thus taking limits as $t \to 0$

$$f'(x)(y - x) \geq f(y) - f(x)$$

For the case of $\mathbb{R}^n$, for x and $y \in U$ and $t \in (0, 1)$, define the function

$$g(t) = f(ty + (1-t)x)$$

and note that

$$\begin{aligned} g(t) &= f(ty+(1-t)x) \\ &= f(x_1+t(y_1-x_1), x_2+t(y_2-x_2), ..., x_n+t(y_n-x_n)) \end{aligned}$$

so, by the chain rule

$$g'(t) = \frac{\partial f(ty+(1-t)x)}{\partial t} = \sum f_{x_i}(ty+(1-t)x)(y_i-x_i)$$

so

$$g'(0) = \sum f_{x_i}(x)(y_i-x_i)$$

Now we need to prove that f is concave if and only if g is concave. To see this note that, for any $s, t \in [0,1]$

$$\begin{aligned} g(\alpha s+(1-\alpha)t) &= \\ & f((\alpha s+(1-\alpha)t)\, y+(1-\alpha s-(1-\alpha)t)x) \\ &= f(\alpha(sy+(1-s)x)+(1-\alpha)(ty+(1-t)x) \\ &\geq \alpha f(sy+(1-s)x)+(1-\alpha)f(ty+(1-t)x) \\ &= \alpha g(s)+(1-\alpha)g(t) \end{aligned}$$

Similarly, if g is concave, then

$$\begin{aligned} f(ty+(1-t)x) &= g(t) = g(t1+(1-t)0) \\ &\geq tg(1)+(1-t)g(0) \\ &= tf(x)+(1-t)f(y) \end{aligned}$$

Thus, by the previous result, we know that

$$\begin{aligned} g(t)-g(s) &\leq g'(s)(t-s) \\ &\Rightarrow g(1)-g(0) \leq g'(0)(1-0) \\ &\Rightarrow f(y)-f(x) \leq \sum f_{x_i}(x)(y_i-x_i) \end{aligned}$$

Question 2 (35 Points) Let $u : \mathbb{R}^n_+ \to \mathbb{R}$ be a continuous utility function on an n-dimentional commodity space, and $B : \mathbb{R}_{++} \rightrightarrows \mathbb{R}^n$ be the budget constraint defined by $B(p) = \{x \in \mathbb{R}^n_+ | px \leq I\}$ for some I

1. Consider the problem of a consumer that has to spend at least k_i on each good i. Write down an expression for the choice set of the consumer as a function of p. Call this set $B^*(p)$. Write down conditions on the k_i's that ensures that the choice set is non-empty. For the remainder of the question, assume that this condition holds

   Answer

$$B^*(p) = \left\{ x \in \mathbb{R}^n_+ | px \leq I \text{ and } x_i \geq \frac{k_i}{p_i} \right\}$$

   $\sum_i k_i \leq I$ must hold for the choice set to be non-empty

2. Let $v : \mathbb{R}^n_{++} \to \mathbb{R}$ be defined as $v(p) = \max_{x \in B^*(p)} u(p)$ and $d : \mathbb{R}^n_{++} \Rightarrow \mathbb{R}$ be defined as $d(p) = \arg\max_{x \in B^*(p)} u(p)$. Can we guarantee that $v(p)$ and $d(p)$ are well defined? Is d guaranteed to be a function? What about if the consumer had to spend more that k_i on each good? In either case prove that the v and d are well defined, or give an example where they are not

   Answer: $B^*(p)$ is closed (as it is intersection of a finite number of closed sets, $B(p)$ and $\{x \in \mathbb{R}^n | x_i \geq \frac{k_i}{p_i}\}$) and bounded for any p, and u is continuous, so we can guarantee that by Weierstrass theorem, v and d are well defined. d is not guaranteed to be a function, it could be a correspondence (for example, let $n = 2.u(x) = x_1 + x_2$ $k_1 = k_2 = 0$, $p_1 = p_2 = 1$). In the latter case, v and d may not be well (for example, let $n = 2.u(x) = x_1$ $k_1 = k_2 = 0$, $p_1 = p_2 = 1$)

3. Assume once again that the consumer has to spend at least k_i on each good. Can we conclude that $v(p)$ is continuous? what about $d(p)$? Again, either prove that they are, or give an example in which they are not

   Answer: We can conclude that $v(p)$ is continuous because the conditions of the theorem of the maximum hold: We have shown that $B^*(p)$ is compact valued for every p. We need to show that it is continuous. One easy way to see this is to note that $B^*(p)$ is the intersection of $n+1$ constraints: the budget constraint $B(p)$ and n constraints that $x_i \geq \frac{k_i}{p_i}$. We might assume that. because each of these correspondences is continuous in p, then their intersection is also continuous. and this is in fact the case. For notational convenience, define $K_i(p) = \left\{x \in X | x_i \geq \frac{k_i}{p_i}\right\}$, and note that $B^*(p) = B(p) \cap_{i=1}^n K_i(p)$. To prove LHC, take some open set O and $p \in \mathbb{R}^n_{++}$ such that $B^*(p) \cap O \neq \emptyset$. This implies that $B(p) \cap O \neq \emptyset$ and $K_i(p) \cap O \neq \emptyset$ for any i. As all of these correspondences are continuous, this implies that for each set there is a δ_i such that, for each constraint and $p' \in B(p, \delta_i)$

$$K_i(p') \cap O \neq \emptyset$$

and a δ_B such that, for $p' \in B(p, \delta_B)$

$$B(p) \cap O \neq \emptyset$$

setting δ equal to $\min\{\{\delta_i\}_{i=1}^n \delta_B\}$ tells us that, for all $p' \in B(p, \delta_i)$

$$B^*(p) \cap O \neq \emptyset$$

For upper hemi continuity we can make a sequence argument. Let $p_m \to p$, and take a sequence $y_m \in B^*(p_m)$. Clearly,$y_m \in B(p_m)$, and as this is UHC and compact valued, there exists a subsequence y_m^1 that converges to a point within $B(p)$. In turn, this subsequence lies inside $K_1(p_m)$ which while not compact valued is closed. and we know that y_m^1 is bounded. Thus, there is a subsequence y_m^2 that converges to a point in $K_1(p)$, and so $K_1(p) \cap B(p)$ iterating on this procedure tells us that there is a sequence y_m^{n+1} that converges to a point in $B^*(p)$ and so we are done.

u is continuous and there is always a feasible choice from any $B^*(p)$ so the theorem of the maximum holds, and $v(p)$ is continuous.

$d(p)$ is not necessarily a continuous correspondence however. Let $n = 2.u(x) = x_1 + x_2$, $k_1 = k_2 = 0$, and consider the sequence p_m where $p_m^1 = 1$ and $p_m^2 = 1 + \frac{1}{m}$.so $p_m \to (1, 1)$ Here $d(p_m) = (I, 0)$ for every p_m (i.e. it is optimal to spend all income on good 1), but $d(p) = \{x_1, x_2 | x_1 + x_2 = I\}$. (i.e. any way of spending all income is optimal). Thus, d is not lower hemi continuous

4. Now say that the consumer only has to spend k_i on good i if the price of i is below some level $\bar{p}_i$ (i.e. if the price of good i goes above $\bar{p}_i$,then there is no restriction on how much has to be spent on i). Can we guarantee that v is continuous?

   Answer: No, v need not be continuous now, as $B^*(p)$ is not continuous. Let $n = 2.u(x) = x_1$, $k_1 = 0$, $k_2 = 1$ and $\bar{p}_2 = 1$ and $I = 2$. Then consider the sequence $p_m = (1, 1 - \frac{1}{m}) \to (1, 1)$. For p_m, the consumer has to spend 1 on good 2, so $v(p_m) = 1$. However, for p, this constraint is lifted, so $p = 2$

5. Now forget the additional condition in section 4 (i.e. return to the conditions in section 3). Let $n = 2$, and assume that $u(x_1, x_2) = x_1^\alpha x_2^\beta$. Under what conditions on α and β can

we conclude that the KKT conditions are necessary and sufficient for a local maximum? Under these conditions will any local maximum also be a global maximum? Assume that these conditions hold, and fix p such that $p_1 = p_2 = 1$ and $I = 2$. For any pair of parameters α and β, find conditions on k_1 and k_2 such that $\arg\max_{x \in B^*(p)} u(p) = \arg\max_{x \in B(p)} u(p)$

Answer: Note that the constraint set is convex for any p, so as long as $u(x_1, x_2)$ is concave, we know that the KKT conditions are necessary and sufficient. In fact, this function is concave. We can note this by looking at the Hessian:

$$H(u(x_1, x_2)) = \begin{bmatrix} \alpha(\alpha-1)x_1^{\alpha-2}x_2^{\beta} & \alpha\beta x_1^{\alpha-1}x_2^{\beta-1} \\ \alpha\beta x_1^{\alpha-1}x_2^{\beta-1} & \beta(\beta-1)x_1^{\alpha}x_2^{\beta-2} \end{bmatrix}$$

Now, the first order principle minors (which we want to be negative) are

$$\alpha(\alpha-1)x_1^{\alpha-2}x_2^{\beta}$$
$$\beta(\beta-1)x_1^{\alpha}x_2^{\beta-2}$$

while the second order principle minor, which we want to be positive is given by

$$\alpha(\alpha-1)x_1^{\alpha-2}x_2^{\beta}\beta(\beta-1)x_1^{\alpha}x_2^{\beta-2} - \alpha\beta x_1^{\alpha-1}x_2^{\beta-1}\alpha\beta x_1^{\alpha-1}x_2^{\beta-1}$$

So as long as $\alpha \in (0, 1)$ and $\beta \in (0.1)$ this is a convex programming problem and the KKT first order conditions are sufficient. This also implies that any local maxima will also be global maxima.

One way to deal with the second portion of the question is to solve the (relatively) unconstrained problem of maximizing with respect to $B(p)$, and finding conditions under which this gives solutions such that the additional constraints are satisfied. Therefore the Lagrangian becomes

$$L(x, \lambda) = x_1^{\alpha}x_2^{\beta} + \lambda(x_1 + x_2 - 2)$$

giving FOC

$$\begin{aligned} \alpha x_1^{\alpha-1}x_2^{\beta} &= -\lambda \\ \beta x_1^{\alpha}x_2^{\beta-1} &= -\lambda \\ x_1 + x_2 &= 2 \end{aligned}$$

The first two conditions give

$$\frac{\alpha}{\beta}\frac{x_2}{x_1} = 1$$

or

$$x_2 = \frac{\beta}{\alpha}x_1$$

substituting into the budget constraint gives

$$x_1 + \frac{\beta}{\alpha}x_1 = 2$$

and so

$$x_1^* = \frac{\alpha}{\alpha+\beta}2$$

similarly,

$$x_2^* = \frac{\beta}{\alpha+\beta}2$$

Thus, for this also to be the solution of the (more) constrained problem, we need

$$\begin{aligned} p_1 x_1^* &\geq k_1 \\ &\Rightarrow \frac{\alpha}{\alpha+\beta}2 \geq k_1 \end{aligned}$$

and

$$\begin{aligned} p_2 x_2^* &\geq k_2 \\ &\Rightarrow \frac{\beta}{\alpha+\beta}2 \geq k_1 \end{aligned}$$

Question 3 (20 Points) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a concave and continuously differentiable function, and $g_j : \mathbb{R}^n \to \mathbb{R}$, $j = 1, ..., m$, be convex and continuously differentiable functions. Consider the optimization problem (P):

$$\begin{aligned} &\max (f(x)) \\ &s.t. \\ g(x) &\leq 0 \end{aligned}$$

Let $X = \mathbb{R}^n$, $U = \mathbb{R}^n_-$ and $l : X \times U \to R$ be the (Lagrangian) function

$$l(x, \mu) = f(x) + \Sigma_j \mu_j g_j(x)$$

A saddle point for l is a vector $(x^*, \mu^*) \in X \times U$ that satisfies

$$l(x, \mu^*) \leq l(x^*, \mu^*) \leq l(x^*, \mu)$$

for all $(x, \mu) \in X \times U$.

1. Show that if (x^*, μ^*) is a saddle point for l then x^* is an optimal solution for (P).

   Answer: First of all, we need to show that x^* is feasible. To see this, note that the restriction

$$l(x^*, \mu^*) \leq l(x^*, \mu)$$

Implies that

$$\Sigma_j \left(\mu_j^* - \mu_j\right) g_j(x^*) \leq 0$$

for all $\mu_j \leq 0$. So, for any j, let $\bar{\mu} = \left(\mu_1^*, ...\mu_j^* - 1, ...\mu_m^*\right)$, which, when applied to the equation above, gives $g_j(x^*) \leq 0$. Now also note that, setting μ to zero, the above equation gives

$$\Sigma_j \mu_j^* g_j(x) \leq 0$$

And, as $\mu_j^* \leq 0$ and $g_j(x) \leq 0 \; \forall \; j$, we get $\mu_j^* g_j(x) \geq 0 \; \forall \; j$, and so

$$\Sigma_j \mu_j^* g_j(x) \geq 0$$

which in turn implies that

$$\Sigma_j \mu_j^* g_j(x) = 0$$

Thus, we have that

$$\begin{aligned} l(x, \mu^*) &\leq l(x^*, \mu^*) \\ &\Rightarrow f(x) + \Sigma_j \mu_j^* g_j(x) \leq f(x^*) + \Sigma_j \mu_j^* g_j(x^*) \\ &\Rightarrow f(x) + \Sigma_j \mu_j^* g_j(x) \leq f(x^*) \end{aligned}$$

and so, as $\Sigma_j \mu_j^* g_j(x) \geq 0$ for any feasible x (so that $g_j(x) \leq 0$),

$$f(x) \leq f(x^*)$$

2. Conversely, assume that (P) satisfies the Slater constraint qualification condition (SCQ) ; there exists $\overline{x}$ such that $g(\overline{x}) < 0$. Show that if x^* solves (P) then there exists a μ^* such that (x^*, μ^*) is a saddle point for l (hint, use the answer to question 1, and think about the KKT conditions)

   Answer: If the Slater constrain qualification holds, we know that if x^* is a maximum, then it satisfies the KKT necessary conditions, and so

$$\frac{\partial l(x^*, \mu^*)}{\partial x_i} = \frac{\partial f(x^*)}{\partial x_i} + \Sigma_j \mu_j^* \frac{\partial g_j(x^*)}{\partial x_i} = 0$$

   Note that, as $\mu_j^* \leq 0 \; \forall \; j$, $l(.)$ is a concave function, so by the result of question 1

$$l(x, \mu^*) - l(x^*, \mu^*) \leq \sum_{i=1}^{n} \frac{\partial}{\partial x_i} l((x^*, \mu^*)(y_i^* - x_i^*) = 0$$

   giving

$$l(x, \mu^*) \leq l(x^*, \mu^*)$$

   On the other hand, by the complimentary slackness condition, $\Sigma_j \mu_j^* g_j(x^*) = 0$, while, for any other $\mu \leq 0$, given that $g_j(x^*) \leq 0$ we have $\Sigma_j \mu_j g_j(x^*) \geq 0$ and so

$$\begin{aligned} l(x^*, \mu^*) &= f(x^*) + \Sigma_j \mu_j^* g_j(x^*) \\ &\leq f(x^*) + \Sigma_j \mu_j g_j(x^*) \\ &= l(x^*, \mu) \end{aligned}$$

Question 4 (30 Points) An affine hull is defined as follows: For a set B in a linear space V, the affine hull of B is the smallest affine manifold that contains B.

1. Show that this means that

$$aff(B) = \cap \{M \subset V | M \text{ is an affine manifold and } B \subset M\}$$

   Answer: The key thing to note is that the union of an arbitrary number of affine manifolds is itself an affine manifold. Probably the easiest way to see this is using the fact that S is an affine manifold if and only if $\lambda x + (1 - \lambda) y \in S$ for all $x, y \in S$, $\lambda \in \mathbb{R}$. Thus, if $\mathcal{A}$ is a set of affine manifolds, $M = \cup_{A \in \mathcal{A}} A$ and $x, y \in M$, then $x, y \in A \; \forall \; A \in \mathcal{A}$. Thus, $\lambda x + (1 - \lambda) y \in A \; \forall \; A \in \mathcal{A}$, and thus $\lambda x + (1 - \lambda) y \in M$. Thus the set $\cap \{M \subset V | M \text{ is an affine manifold and } B \subset M\}$ is itself an affine manifold and contains B. It is also trivially the smallest such manifold.

2. Show that this also means that

$$aff(B) = \left\{ \sum_{i=1}^{k} \lambda_i x^i | k \in \mathbb{N},\, x_1, .., x_n \in B \text{ and } \sum_{i=1}^{k} \lambda_i = 1 \right\}$$

Hint - you can use the result that a set S is an affine manifold if and only if $\lambda x + (1-\lambda) y \in S$ for all $x, y \in S$, $\lambda \in \mathbb{R}$

Answer: We can prove that

$$aff(B) \supset \left\{ \sum_{i=1}^{k} \lambda_i x^i | k \in \mathbb{N},\, x_1, .., x_n \in B \text{ and } \sum_{i=1}^{k} \lambda_i = 1 \right\}$$

by induction on the size k. Clearly it is true for $k = 1$, now assuming it is true for k, note that

$$\begin{aligned} x &= \sum_{i=1}^{k} \lambda_i x^i \\ &= \lambda_1 x^1 + (1-\lambda_1) \sum_{i=2}^{k} \frac{\lambda_i}{(1-\lambda_1)} x^i \end{aligned}$$

As $\sum_{i=2}^{k} \frac{\lambda_i}{(1-\lambda_1)} = 1$, by the inductive hypothesis $\bar{x} = \sum_{i=2}^{k} \frac{\lambda_i}{(1-\lambda_1)} x^i \in aff(B)$. thus,

$$\begin{aligned} &= \lambda_1 x^1 + (1-\lambda_1) \sum_{i=2}^{k} \frac{\lambda_i}{(1-\lambda_1)} x^i \\ &= \lambda_1 x^1 + (1-\lambda_1) \bar{x} \end{aligned}$$

where both x^1 and $\bar{x}$ are in $aff(B)$. As $aff(B)$ is an affine manifold, by the result given in the hint, this implies that

$$\lambda_1 x^1 + (1-\lambda_1) \bar{x} \in aff(B)$$

Now all we have to show is that

$$aff(B) \subset \left\{ \sum_{i=1}^{k} \lambda_i x^i | k \in \mathbb{N},\, x_1, .., x_n \in B \text{ and } \sum_{i=1}^{k} \lambda_i = 1 \right\}$$

but, as the set on the right clearly contains B, all that needs to be shown is that it is an affine manifold. Take x, y that are in this set, and note that, for some k, λ^x, λ^y we have

$$\begin{aligned} x &= \sum_{i=1}^{k} \lambda_i^x x^i \\ y &= \sum_{i=1}^{k} \lambda_i^y y^i \end{aligned}$$

for $\sum_{i=1}^k \lambda_i^x = 1$, $\sum_{i=1}^k \lambda_i^y = 1$ and $\{x^i\} \in B$, $\{y^i\} \in B$, .Thus

$$\begin{aligned} z &= tx + (1-t)y \\ &= t\sum_{i=1}^k \lambda_i^x x^i + (1-t)\sum_{i=1}^k \lambda_i^y y^i \\ &= \sum_{i=1}^k t\lambda_i^x x^i + \sum_{i=1}^k (1-t)\lambda_i^y y^i \end{aligned}$$

but, as $\sum_{i=1}^k t\lambda_i^x + \sum_{i=1}^k (1-t)\lambda_i^y$, this implies that z is in the set, and so it is an affine manifold

3. Calculate the affine manifolds of the following sets in $\mathbb{R}^3$

   (a) $S = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}$

   (b) $S = \{x | x_1 = 1 \text{ and } (x_2 - 1)^2 + (x_2 - 1)^2 \leq 1\}$

   (c) $S = \{x | x_1^2 + x_2^2 + x_3^2 \leq 2\}$

   Answer. Remember that in $\mathbb{R}^3$, we have four types of affine manifolds: points, lines planes, and $\mathbb{R}^3$ itself. For a, the set is an affine manifold, and so is its own affine hull. For b, clearly S is contained in the plane described by $x_1 = 1$. Equally clearly, no line can contain S, so this plane is the affine hull. For c, $\mathbb{R}^3$ is the affine hull, as no plane can contain S.

4. We define the concept of the relative interior of a set as follows: Let S a subset of a linear space V. A vector $x \in S$ is called a relative interior point of S if, for any $y \in aff(S)$ there exists a $\alpha_y > 0$ such that

   $$(1-\alpha)x + \alpha y \in S \text{ for all } 0 \leq \alpha \leq \alpha_y$$

   The set of all relative interior points of S is called the relative interior of S. We denote this $ri(S)$

   (a) Let S be a set in $\mathbb{R}^n$. Show that, if $x \in int(S)$ then $x \in ri(S)$

   (b) Find an example of a set S and a point $x \in ri(S)$ but not $x \in int(S)$

   (c) Calculate the relative interiors and interiors for the sets in part

   Answer: Assume $x \in int(S)$, but for some $y \in aff(S)$, there exists no $\alpha_y > 0$ such that $(1-\alpha)x + \alpha y \in S$ for all $0 \leq \alpha \leq \alpha_y$. In this case, we could construct a sequence

$\alpha_n \to 0$ such that

$$(1-\alpha_n)x + \alpha y \notin S$$

for all α_n. But this sequence converges to x, implying that there is no ε such that $B(x, \varepsilon) \in S$, a contradiction

For a counterexample for part b, consider S in part a above. This is clearly in the relative interior of S (as it is its own affine hull), but is not an interior point.

For the three sets above, the relative interior in part a is S itself. In part b it is $\{x | x_1 = 1 \text{ and } (x_2-1)^2 + (x_2-1)^2 < 1\}$ and in part c it is the same as the interior of the set

5. Prove the following: Let C be a convex set in some metric space M. Let $x_1 \in ri(C)$ and $x_2 \in cl(C)$. Then $[x_1, x_2) \subset ri(c)$

   Answer: Consider an affine hull $A = aff(C)$ as a metric subspace of the space $\mathbb{R}^n$. Furthermore it is closed in $\mathbb{R}^n$. Therefore, $cl(X)$ relative to A equals $cl(X)$ relative to $\mathbb{R}^n$ for all $X \subset A$. Also, by definition, $ri(X)$ equals $int(X)$ relative to A for all $X \subset A$. Therefore, the result we proved in class that this statement is true for $x_1 \in int(C)$ can be used to prove this result.