Linear Algebra

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Lecture Notes for Fall 2014 PhD Class - Brown University

1 Lecture 1[1]

1.1 Definition of Linear Spaces

In this section we define the concept of a *linear* (or vector) space. The basic ingredients of a linear space are a collection of objects, and a definition of two operations: addition and scalar multiplication that obey certain properties:

Definition 1 *Let V be a non- empty set. The list $(V, +, \bullet)$ is a* ***linear space*** *if $+$ is a binary operation on V and $\bullet$ is a mapping that assigns each $(\lambda, v) \in \mathbb{R} \times V$ an element $\lambda \bullet v$ of V (which we denote λv)* ***such that*** *for any $\alpha, \lambda \in \mathbb{R}$ and $v, w, z \in V$ the following properties hold:*

1. *Additive properties*
   (a) *(associativity) $(x + y) + z = x + (y + z)$*
   (b) *(existence of a zero element) There exists an element $\emptyset \in X$ such that $\emptyset + x = x = x + \emptyset$ for all $x \in X$*
   (c) *(existence of inverse elements) For each $x \in X$, there exists an element $-x \in X$ such that $x + -x = \emptyset = -x + x$.*
   (d) *(Commutativity) $x + y = y + x$ for all $x, y \in X$*
2. *Scalar multiplication properties*

[1] For another guide to the material in these first two lectures, see Ok Chapter F

(a) *(associativity)* $\alpha(\lambda x) = (\alpha\lambda)x$

(b) *(distributivity)* $(\alpha + \lambda)x = \alpha x + \lambda x$ *and* $\lambda(x + y) = \lambda x + \lambda y$

(c) *(The unit rule)* $1x = x$.

Thus, a linear space is a set of objects and a pair of operations that satisfy our 'intuitive' notion of how addition and scalar multiplication work. An obvious example of a linear space (which you will prove in the homework) is $\mathbb{R}^N$, with addition and scalar multiplication defined in the usual way (we will write $x \in \mathbb{R}^N$ as $x = (x_1, .., x_n)$ with x_i being the ith element of the vector x):

$$x + y = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_i + y_i \\ \vdots \\ x_n + y_n \end{pmatrix} \qquad \lambda x = \begin{pmatrix} \lambda x_1 \\ \vdots \\ \lambda x_i \\ \vdots \\ \lambda x_n \end{pmatrix}$$

There are plenty of other examples of linear spaces based on objects outside $\mathbb{R}^N$

Example 1 *Let V be the set of polynomials of degree less than or equal to n defined on $[0, 1]$. In other words, $p \in V$ implies that $p : [0, 1] \to \mathbb{R}$ and $p(x) = \sum_{i=0}^{n} \alpha_i x^i$ for some collection of scalars $\alpha_0, ...\alpha_n$. Let $p, q \in V$ be two arbitrary elements defined by $p(x) = \sum_{i=0}^{n} \alpha_i x^i$ and $q(x) = \sum_{i=0}^{n} \beta_i x^i$ and define the binary operation $\oplus$*

$$p \oplus q = \sum_{i=0}^{n} (a_i + \beta_i) x^i$$

and let multiplication, $\odot$, be defined as

$$\lambda \odot p = \sum_{i=0}^{n} (\lambda \alpha_i) x^i$$

Then $(V, \oplus, \odot)$ is a linear space

Proof. *Left as an exercise* ■

Why are linear spaces important? Well, a linear structure is one of the basic mathematical structures that embody $\mathbb{R}^N$ That is, $\mathbb{R}^N$ has a natural or intuitive linear structure. The notion of a linear space formalizes these properties, and building off them we are going to be able to derive a number of results in functional and convex analysis, which are in turn going to be useful in all sorts of branches of economics - static and dynamic optimization, welfare theorems, utility representations and the like. However, there is nothing 'special' about $\mathbb{R}^N$, in the sense that it is just one example of a linear space (as we have seen). By working in the general class of linear space, then everything we prove will hold true not just for $\mathbb{R}^N$, *but for any linear space.* This makes the results we show much more powerful and elegant. In terms of power, it means our results hold for (for example) the space of polynomial functions defined above, which can come in handy in dynamic optimization and econometric theory In terms of elegance, we are making explicit exactly *what* properties of $\mathbb{R}^N$ that are allowing us to derive these results

Our first theorem elucidates some basic properties of linear spaces

Theorem 1 *Let* $(V, +, \bullet)$ *be a linear space. Then*

1. *The zero vector is unique*
2. *each* $x \in V$ *has a unique inverse*
3. $0x = \emptyset \ \forall \ x \in V$
4. $-x = (-1)x \ \forall \ x \in V$
5. $r\emptyset = \emptyset \ \forall \ r \in \mathbb{R}$

Proof.

1. *Take two elements* $\emptyset$, $\hat{\emptyset}$ *such that* $\emptyset + x = x = x + \emptyset$ *and* $\hat{\emptyset} + x = x = x + \hat{\emptyset}$ *for all* $x \in X$. *Then* $\emptyset + \hat{\emptyset} = \hat{\emptyset} = \hat{\emptyset} + \emptyset = \emptyset$.
2. *Let* w, z *be inverses of* x. *Then*

$$\begin{aligned} & w \\ = \ & w + (x + z) \\ = \ & (w + x) + z \\ = \ & z \end{aligned}$$

Where the second line follows from the definition of the inverse (which implies that $(x+z)=\emptyset$*) and the definition of the zero element (which implies that* $w+\emptyset=w$*). The third line follows from the associativity property, and the fourth line once again follow from the definition of an inverse and zero element.*

3. *Let* $x, y \in V$ *be two arbitrary elements. Then*

$$
\begin{aligned}
(0+1)x &= 0x+1x \\
&\Rightarrow\ x=0x+x \\
&\Rightarrow\ x+(-x+y)=0x+x+(-x+y) \\
&\Rightarrow\ (x+(-x))+y=0x+(x+(-x))+y \\
&\Rightarrow\ y=0x+y
\end{aligned}
$$

Where the first line follows from distributivity, the second from the unit rule, the third definitionally, the fourth from associativity and the fifth from the definition of the inverse. As y *was chosen arbitrarily, this is enough to confirm that* $0x$ *must be the zero vector, and as* x *was chosen arbitrarily, this must be true of all* $x \in \dot{V}$

4. *Left as exercise*

5. *Left as exercise*

■

A linear subspace is a subset of a linear space

Definition 2 *Let* $(V,+,\bullet)$ *be a linear space, and* $W \subset V$*. We say that* W *is a linear subspace if* $(W,+,\bullet)$ *is a linear space.*

The key to checking whether W is a linear subspace is to check whether it is closed under addition and scalar multiplication.

Remark 1 *Let* $(V,+,\bullet)$ *be a linear space. The set* $W \subset V$ *is a linear subspace if and only if it is closed under addition and scalar multiplication (i.e. for any* $x, y \in W$ *and* $\lambda \in \mathbb{R}$*,* $x+y \in W$ *and* $\lambda x \in W$*)*

Proof. *Remember that W is a linear subspace if $(W, +, \bullet)$ is a linear space. For this to be true, it must be that $+$ maps $W \times W \to W$ and $\bullet$ maps $\mathbb{R} \times W \to W$. The 'only if' part of the claim is therefore obvious, as is the fact that this condition is satisfied if W is closed under addition and scalar multiplications. In terms of the remaining seven properties, the only two that are not immediate (you should check) are the existence of a zero element and the existence of an inverse element. However, The proof of theorem 1 sections 3 and 4 shows us that (i) $0x = \emptyset \ \forall \ x \in W$ and (ii) $-x = (-1)x \ \forall \ x \in W$. Thus, closure under scalar multiplication guarantees these two properties* ■

Consider the following two examples

Example 2 *We know that $(\mathbb{R}^2, +, \bullet)$ is a linear space when $+$ and $\bullet$ are defined in the usual way. Now consider $W = \{x \in \mathbb{R}^2 | x_1 = x_2\} \subset \mathbb{R}^2$. We claim that W is a linear subspace of $(\mathbb{R}^2, +, \bullet)$. From remark 1 we know that all we have to show is that W is closed under addition and scalar multiplication. First addition. Take $x, y \in W$*

$$\begin{aligned} & x + y \\ = & \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \\ = & \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix} \end{aligned}$$

but, as $x, y \in W$, we know that $x_1 = x_2$ and $y_1 = y_2$. Thus $x_1 + y_1 = x_2 + y_2$ and so $x + y \in W$. Next, scalar multiplication. $\lambda x = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \end{pmatrix}$, so again, if $x_1 = x_2$ then $\lambda x_1 = \lambda x_2$ and $\lambda x \in W$

Example 3 *Again, starting with $(\mathbb{R}^2, +, \bullet)$ as a linear space, $W = [0, 1]^2 \subset \mathbb{R}^2$ is NOT a linear subspace, because it is not closed under addition or scaler multiplication (consider $\begin{pmatrix} 0.75 \\ 0.75 \end{pmatrix} + \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$)*

Next,we introduce the concept on **linear dependence** (and independence). First we need to introduce the idea of a **linear combination**

Definition 3 *Let $(V, +, \bullet)$ be a linear space, and $\{x_1, ..., x_n\} \subset V$ be a finite subset of V. A linear combination of $\{x_1, ..., x_n\}$ is an element $x \in V$ such that*

$$x = c_1 x_1 + ... + c_n x_n$$

for some collection $\{c_1, ..., c_n\} \in \mathbb{R}^n$

A linear combination of a set of elements in V is therefore any other element in V that can be achieved using the elements of that set and the operations of addition and scalar multiplication finitely many times.

Intuitively, a set of objects is linearly dependant if one of its elements can be constructed using its other elements

Definition 4 *Let $(V, +, \bullet)$ be a (non-trivial) linear space. A subset $S \subset V$ is linearly dependent if at least one element $x \in S$ can be expressed as a linear combination of finitely many elements of $S/\{x\}$. S is linearly independent if no finite subset of it is linearly dependent.*

For finite subsets of $\{x_1, ..., x_n\} \subset V$, an equivalent statement is that $\{x_1, ..., x_n\}$ is linearly independent if and only if

$$\emptyset = c_1 x_1 + ... + c_n x_n$$

only if $c_i = 0 \; \forall \; i$. It is straightforward to show this equivalence, but you should check to make sure you understand the concept.

In $\mathbb{R}^2$, the concept of linear dependence is intuitive: any two vectors x y are linearly dependent if and only if they lie on the same line through the origin (check that you agree with this statement. It follows from the fact that, if x and y are linearly dependent, then it must be the case that $x = \lambda y$ for some $\lambda \in \mathbb{R}$, implying that $x_2 = \frac{y_2}{y_1} x_1$, which is the slope of the line through the origin and y)

What about three vectors in $\mathbb{R}^2$? well, it turns out that it is impossible to have a collection of three linearly independent vectors in $\mathbb{R}^2$. This we can show geometrically, and you will prove for homework. This is a demonstration of a much more general property that we will come onto in the next lecture.