Mathematics For Economists

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Midterm

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DON'T PANIC!

NOTE: Prove all statements, unless told otherwise!

Question 1 Consider the set of functions $F \subset \mathcal{C}(0,1)$ defined by

$$F = \left\{ f|\mathcal{C}(0,1)]|f(x) = ax^b \text{ , } a \in A \subset \mathbb{R} \text{ and } b \in B \subset \mathbb{R} \right\}$$

. That is, F is a subset of the set of continuous functions defined on $(0,1)$.

1. (5pts) Let $B = \mathbb{R}$. Is every element of this set of functions continuous? Are they all uniformly continuous? Are they all Lipschitz continuous? (HINT: for the last part, you can use the fact that $\frac{f(x)-f(y)}{x-y}$ converges to $f'(x)$ as y converges to x)
2. (5pts) Does your answer change if we instead have $B = \mathbb{R}_{++}$?

   Now assume that $B \subset \mathbb{R}_{++}$ and that $F \subset \mathcal{C}[0,1]$, which we endow with the sup metric (i.e. $d(f,g) = \sup_{x \in [0,1]} |f(x) - g(x)|$)
3. (3 pts) Can we ensure that the distance between any two functions in F is finite?
4. (7pts) Is F bounded? If not, provide necessary and sufficient conditions on A and B for it to be bounded?
5. (10pts) Is F complete? If not, provide sufficient conditions on A and B for it to be complete (you will get more points if you can name looser conditions!)
6. (5pts) Define $T : F \to \mathbb{R}$ as $T(f) = \max_{x \in [0,1]} f(x)$. Is T well defined (i.e. does $T(f)$ take a value in $\mathbb{R}$ for any $f \in F$)?

7. (15pts) Provide conditions under which the problem $\max_{f \in F} T(f)$ has a solution (again, you will get more points for looser conditions!)

Question 2 (20 pts) Let X be a metric space such that there exists an $\varepsilon > 0$ and an uncountable set $S \subseteq X$ such that $d(x, y) > \varepsilon$ for any distinct $x, y \in S$. Show that X cannot be separable. Give an example of three spaces that satisfy this property.

Question 3 Consider the following subsets of the set of all infinite sequence

$$\begin{aligned} \ell^p &= \left\{ \{x\}_m \in \mathbb{R}^\infty | \sum_{i=1}^{\infty} |x_i|^p < \infty \right\} \\ \ell^\infty &= \left\{ \{x\}_m \in \mathbb{R}^\infty | \sup_{m \in \mathbb{N}} |x_m| < \infty \right\} \end{aligned}$$

1. (5 pts) Show that ℓ^1 is a strict subset of ℓ^∞
2. (10 pts) Is ℓ^1 a linear subspace of ℓ^∞?
3. (15 pts) Is the set of all convergent real sequences a linear subspace of ℓ^∞? or of ℓ^1?