MATHEMATICS FOR ECONOMISTS

Midterm - 26 th October
Suggested Solution

Question 1.-

Let $L : \mathbb{R} \to \mathbb{R}$ be a linear functional on $\mathbb{R}$. Is it necessarily the case that L maps open sets to open sets? If so, prove it. If not, provide a counterexample, and a condition on L such that the statement is true. What about if L is a linear operator mapping $\mathbb{R}^2$ to $\mathbb{R}^2$.(hint, what must L($\mathbb{R}^2$) look like?

Definition *Let V and W be two linear spaces. A function $L : V \to W$ is called a linear operator if $L(\alpha x + x') = \alpha L(x) + L(x')$ for all $x, x' \in V, \alpha \in \mathbb{R}$. A real valued linear operator is called a linear function*

Remark *(7) The function $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear operator if and only if $\exists A \in \mathbb{R}^{m \times n}$ such that $L(x) = Ax$ for all $x \in \mathbb{R}^n$*

(a) Let $L : \mathbb{R} \to \mathbb{R}$ be a linear functional on $\mathbb{R}$. Is it necessarily the case that L maps open sets to open sets?.

No it is not necesary. Let consider the linear functional given by $L(x) = 0x = 0$ for all $x \in \mathbb{R}$. The $L(x)$ is a linear functional but it maps any set, in particular open sets, to the closed set given by $\{0\}$. Therefore the condition on L so the statment is true is that the associated matrix is full rank, in this case the matrix is only one element and we need this element $a \neq 0$.

(b)What about if L is a linear operator mapping $\mathbb{R}^2$ to $\mathbb{R}^2$.(hint, what must L($\mathbb{R}^2$) look like?)

No it is not necesary the case, again assume the linear operator $L(x) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} x = 0 \in \mathbb{R}^2$ for all $x \in \mathbb{R}^2$.

The condition that we need is that the associate matrix must be of full rank (Rank=2). [Why?]

Question 2

Recall that we say that a preference relation $\succcurlyeq$ (not necessarily complete) on a set X is upper semi continuous if the set

$$L_{\succ}(x) = \{y \in X / x \succ y\}$$

is open for every $x \in X$ (where $\succ$ is the asymmetric part of $\succcurlyeq$). It is lower semi continuous if the set

$$U_{\succ}(x) = \{y \in X / y \succ x\}$$

is open for every $x \in X$

Part 1.-Characterize the class of preference relations that are upper semi continuous in the discrete topology.

The discrete metric is such that $d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise} \end{cases}$

We want to see what must satisfy the preference relation such that $L_{\succ}(x) = \{y \in X / x \succ y\}$ is an open set, that is for all $z \in L_{\succ}(x)$ there exists $r > 0$ such that $B(z,r) \subset L_{\succ}(x)$ .We know that under the discrete metric if we take $r = 1/2$, $B(z,r) = \{z\} \subset L_{\succ}(x)$, therefore all class of preference relations are upper semi continuous in the discrete topology.

Part 2. In $\mathbb{R}^n$ with the standard topology, give an example of a complete preference relation that is upper semi continuous but not lower semi continuous, and visa versa.

EXAMPLE OF UPPER SEMI CONTINUOUS BUT NOT LOWER SEMICONTINUOUS

Consider the following preferences on (0, 1)

$$x \succ y \text{ if } x \geq 0.5 \text{ and } y < 0.5$$

$$x \sim y \text{ otherwise}$$

For any x, the set $L_{\succ}(x)$ is open, as it is either the empty set or (0, 0.5). However, the set $U_{\succ}(x)$ is not open, as, for $x < 0.5$ it is [0,5,1).

EXAMPLE OF LOWER SEMI CONTINUOUS BUT NOT UPPER SEMICONTINUOUS

Consider the following preferences on (0, 1)

$$x \succ y \text{ if } x > 0.5 \text{ and } y \leq 0.5$$

$$x \sim y \text{ otherwise}$$

why this preferences are lower semi continuous but not upper semicontinuous?

Part 3. Is the following preference relation on $\mathbb{R}^2$ either upper or lower semi-continuous?

$$\text{(a)} \quad x \succcurlyeq y \quad \textit{iff} \;\; x_1 \geq y_1 \text{ and } x_2 \geq y_2$$

UPPER if $L_{\succ}(x) = \{y \in \mathbb{R}^2 / x \succ y\}$ is open for every $x \in \mathbb{R}^2$

In this case $L_{\succ}(x) = \{y \in \mathbb{R}^2 / x \succ y\} = \left\{y \in \mathbb{R}^2 / x \geq y \text{ and } x \neq y\right\}$ that is $y \in \mathbb{R}^2 / x_1 \geq y_1$ and $x_2 \geq y_2$ and at least one strict, which is not open, therefore preferences are not USC $\left[\text{why is not open?}\right]$

LOWER if $U_{\succ}(x) = \{y \in \mathbb{R}^2 / y \succ x\}$ is open for every $x \in \mathbb{R}^2$

In this case $U_{\succ}(x) = \{y \in \mathbb{R}^2 / y \succ x\} = \left\{y \in \mathbb{R}^2 / y \geq x \text{ and } x \neq y\right\}$ that is $y \in \mathbb{R}^2 / x_1 \leq y_1$ and $x_2 \leq y_2$ and at least one of them strict , which is not open, therefore preferences not are LSC $\left[\text{why is not open?}\right]$

$$\text{(b)} \quad x \succcurlyeq y \quad \textit{iff} \;\; x_1 > y_1 \text{ and } x_2 > y_2$$

UPPER if $L_{\succ}(x) = \{y \in \mathbb{R}^2 / x \succ y\}$ is open for every $x \in \mathbb{R}^2$

In this case $L_{\succ}(x) = \{y \in \mathbb{R}^2 / x \succ y\} = \{y \in \mathbb{R}^2 / x > y\}$ that is $y \in \mathbb{R}^2 / x_1 > y_1$ and $x_2 > y_2$, which is open, therefore preferences are USC $\left[\text{why is it open?}\right]$

LOWER if $U_{\succ}(x) = \{y \in \mathbb{R}^2 / y \succ x\}$ is open for every $x \in \mathbb{R}^2$

In this case $U_{\succ}(x) = \{y \in \mathbb{R}^2 / y \succ x\} = \{y \in \mathbb{R}^2 / y > x\}$ that is $y \in \mathbb{R}^2 / x_1 < y_1$ and $x_2 < y_2$, which is open, therefore preferences are LSC $\left[\text{why is it open?}\right]$

$$\text{(c)} \quad x \succcurlyeq y \quad \textit{iff} \;\; x_1 > y_1 \text{ or } x_1 = y_1 \text{ and } x_2 > y_2$$

UPPER if $L_{\succ}(x) = \{y \in \mathbb{R}^2 / x \succ y\}$ is open for every $x \in \mathbb{R}^2$

In this case $L_{\succ}(x) = \{y \in \mathbb{R}^2 / x \succ y\} = \left\{y \in \mathbb{R}^2 / x_1 > y_1 \text{ or } x_1 = y_1 \text{ and } x_2 > y_2\right\}$ which si not open, therefore preferences are not USC $\left[\text{why is not open?}\right]$

LOWER if $U_{\succ}(x) = \{y \in \mathbb{R}^2 / y \succ x\}$ is open for every $x \in \mathbb{R}^2$

In this case $U_{\succ}(x) = \{y \in \mathbb{R}^2 / y \succ x\} = \left\{y \in \mathbb{R}^2 / x_1 < y_1 \text{ or } x_1 = y_1 \text{ and } x_2 < y_2\right\}$, which is not open, therefore preferences not are LSC $\left[\text{why is not open?}\right]$

Part 4.- We say a real valued function f on $\mathbb{R}$ is upper semi continuous if, for every $\alpha \in \mathbb{R}$, the set $\{x \in \mathbb{R} / f(x) < \alpha\}$ is an open set. Show that a function is upper semi continuous if and only if, for every x and every $x_n \to x$,

$$\limsup_{x_n \to x} f(x_n) \leq f(x)$$

We want to prove that if, for every $\alpha \in \mathbb{R}$, the set $\{x \in \mathbb{R} / f(x) < \alpha\}$ is an open set$\Leftrightarrow$ if and only if, for every x and every $x_n \to x, \limsup_{x_n \to x} f(x_n) \leq f(x)$

"$\Rightarrow$" Let assume that for every $\alpha \in \mathbb{R}$, the set $\{x \in \mathbb{R} / f(x) < \alpha\}$ is an open set. We need to show that for every x and every $x_n \to x, \limsup_{x_n \to x} f(x_n) \leq f(x)$

We want to prove that for every x and every $x_n \to x, \limsup_{x_n \to x} f(x_n) \leq f(x)$, or equivantly that $f(x_n) < f(x) + \varepsilon$ for all ε in a neighborhood of x.

We know that for all for every $\alpha \in \mathbb{R}$, the set $\{x \in \mathbb{R} / f(x) < \alpha\}$ is an open set, therefore we know that there exists an $M > 0$ such that $x_n \in B(x, \varepsilon)$, $f(x_n) < f(x) + \varepsilon$ for all $n \geq M$. $\left[\text{ since the set is open we know that for some } \varepsilon \text{ all } x_n \in B(x, \varepsilon) \text{ is such that } f(x_n) < \alpha\right]$ This means that $\sup\{f(x_n) : n \geq M\} \leq f(x) + \varepsilon$ so $\limsup f(x_n) \leq f(x) + \varepsilon$, since $\varepsilon > 0$ is arbitrary here we obtain the desired result.

"$\Leftarrow$"Let assume that for every x and every $x_n \to x, \limsup_{x_n \to x} f(x_n) \leq f(x)$. We need to show that every $\alpha \in \mathbb{R}$, the set $\{x \in \mathbb{R} / f(x) < \alpha\}$ is an open set

Assume not. Then there exists $\varepsilon > 0$ and x_n such that $d(x, x_n) < \frac{1}{n}$ and $f(x_n) > f(x) + \varepsilon$ for all $n \geq M > 1$, therefore $\limsup f(x_n) > f(x)$, but as it was defined $x_n \to x$ and therefore $\limsup_{x_n \to x} f(x_n) \leq f(x)$. which is a contradiction.

Question 3

Consider the following problem

A farmer owns a field. She is must decide what area of that field to enclose to graze sheep. The cost of 1 meter of fence is 1. For simplicty, let's assume that she can only enclose rectangular areas of land. The more land she encloses, the more sheep she can graze, and she receives income of 2 for every sheep that she grazes on her land. However, by enclosing the land, she takes away land that could otherwise be used for camping by her friends that sometimes come from out of town. These friends have a favourate camping spot, and derive utility based on the distance they are from this favourate spot. Unfortunately, the farmer doesn't know what the favourate spot is. The famer cares about money, but also the happiness of her friends, so her utility is an additive function of her income and the utility of her friends Your job is to write down the farmer's optimization problem, and put conditions on the various elements of the problem in order to guarantee a solution. Here are some things that you might find useful. (1) How does the choice set of the friends change with the enclosure chosen by the farmer? And so how does their utility change?.1 (2) You can use the fact that the expectation operator is continuous (3) rather than thinking of a fixed relationship between area and number of sheep, how about a regime in which you can put as many sheep as you like in a given area, but the more sheep you put in the more likely they are to die.

The farmer is going to decided what part to enclose, that we are assuming is a rectangular area, therefore we can define the enclose land by the top left and bottom right coordinate in $\mathbb{R}^2$, let them be a and b respectively ($a \equiv (x_a, y_a) \in \mathbb{R}^2$ and $b \equiv (x_b, y_b) \in \mathbb{R}^2$) that define the rectangle that must be contained in the field. Therefore the cost of the fence is going to be given by $-2|x_b - x_a| - 2|y_b - y_a|$. Then we know that the sheep that she can graze is an increasing function of the land enclosed, that is $f(|(x_b - x_a)(y_b - y_a)|)$ (where by point (3) we know that $f(|(x_b - x_a)(y_b - y_a)|) \in \mathbb{R}$ even when it doesn´t make sense to cut a sheep in half). Finally she also derives utility from her friends utility, where her friends utility is a decreasing function of the distance from the favorite and they are going to select their spot from the non-enclose area, that is the can chose from $\Gamma(a,b) = \left\{c \in L \subset \mathbb{R}^2 / x_c \notin (x_a, x_b) \text{ and } y_c \notin (y_a, y_b)\right\}$, the friends would select the location such that if $p^* \equiv (x^*, y^*)$ is the preferred one, $p^{**} = \arg\max_{p\in\Gamma(a,b)} U(d(p, p^*))$, and the utility of the friends is a decreasing function of the distance, therefore is an increasing function of -distance.. Given that any distance function is continuous in the metric space endowed with itself as a metric, and that the utility function of the friends is continuous in the distance then we know that the utility function of the friends is a continuous function of p.

(1) FRIENDS PROBLEM

The friends problem is to pick a location from the noneclose ones, to maxime their utility, that is

$$p^{**}(a,b) = \arg\max_{p\in\Gamma(a,b)} U(d(p,p^*))$$

where

$$\Gamma(a,b) = \left\{c \in L \subset \mathbb{R}^2 / x_c \notin (x_a, x_b) \text{ and } y_c \notin (y_a, y_b)\right\}$$

We would like to have a continuous $p^{**}(a,b)$ function $\left[\text{see Farmer´s problem}\right]$, therefore we can invoke the theorem of the maximum. We know that $U(p) \equiv U \circ d$ is continuous, so we only need to prove that Γ is compact valued and continuous.

Γ compact valued is almost for free if we assumed that $\Gamma(a,b) = \left\{c \in L \subset \mathbb{R}^2 / x_c \notin (x_a, x_b) \text{ and } y_c \notin (y_a, y_b)\right\}$ $\left[\text{WLOG we are asuming that } a \text{ is the bottom left and } b \text{ is the top right}\right]$, since we are assuming that they can camp on the boundary (fence) and it's bounded because the total land is bounded. Therefore,

Conditions under which $p^{**}(a,b)$ is guaranteed to be continuous

(*) $U(\cdot)$ continuous

(*) Assumption that they can camp on the fence (***)

So if we can prove that the set $\Gamma(a,b)$ is continuous then we are done. We can prove the continuity from the hint and the sequence characterization of lower hemicontinuity.

Definition *A correspondance $\Gamma : X \to Y$ is upper hemicontinuous at $x \in X$ if for every open subset $O \subset Y$ with $\Gamma(x) \subseteq O$ there exists a $\delta > 0$ such that $\Gamma(B(x,\delta)) \subseteq O$*

Let consider (a,b), such that $\Gamma(x) \subset O'$ for O' open, that means that means that $\Gamma(a,b) = \left\{c \in L \subset \mathbb{R}^2 / x_c \notin (x_a, x_b) \text{ and } y_c \notin (y_a, y_b)\right\} \subset O$, what we need to show is that if we consider small perturbations to (a,b), of the way (a', b') where $a' \in B(a, \delta_a)$ and $b' \in B(b, \delta_b)$, we have that

$\Gamma(a',b') = \left\{c \in L \subset \mathbb{R}^2 / x_c \notin (x_{a'}, x_{b'}) \text{ and } y_c \notin (y_{a'}, y_{b'})\right\} \subset O$ for all such (a',b'). Given that $\Gamma(a,b) \subset O$, and O is open that means that for every $c \in \Gamma(a,b)$ there exists some δ_c such that $B(c,\delta_c) \subset O$, take $\delta = \inf_c \delta_c$, then we know that all $c \in \Gamma(a,b)$, $c' \in B(c,\delta) \in \Gamma(a,b) \subset O$, then by the definition of $\Gamma(a',b')$, we can take $\delta_a = \delta_b = \delta$ (or something smaller) and we would have the desired result.

Lemma *A correspondance $\Gamma : X \to Y$ is lower hemicontinuous at $x \in X$ if and only if, for any sequence $x_m \to x$ in X, and any $y \in \Gamma(x)$, there exist a sequence $y_m \in \Gamma(x_m)$ for all x_m*

Take any sequence $(a_m,b_m) \to (a,b)$, if $\Gamma(a,b) \neq \emptyset$ and $\Gamma(a_m,b_m) \neq \emptyset$ for all m, then there exists a sequence $\{c_m\} \in \mathbb{R}^2$ such that $c_m \in \Gamma(a_m,b_m)$, take for example (if a_m is not in the boundary of the land, otherwise similar but with b_m) the sequence $c_m = (a_m - \varepsilon, b_m - \varepsilon)$, where ε can be define as $\varepsilon \equiv \min_n d(a_m,z)$ where z is the closest point on the limit of the land to the sequence a_m (this is a technicality to assure that c_m is still on the farmer´s land)

Therefore we can conclude that $U(a^{**}(x))$ is a continuous function and therefore, since the expectation operator is continuous we have that $EU(a^{**}(x))$.

(2) FARMER´S PROBLEM

Therefore the objetive function of the farmer is a continuous function (assuming that f is continuous) of her choice $x = (x_a, y_a)$, in a compact set, therefore by Weierstrass there exists x*=argmaxEU(farmer), That is the farmer´s problem is given by

$$
\begin{aligned}
(a^*,b^*) &= \arg \max_{a,b \in L \subset \mathbb{R}^2} EU^{farmer}(a,b) \\
&= \arg \max_{a,b \in L \subset \mathbb{R}^2} -2|x_b - x_a| - 2|y_b - y_a| + 2f(|(x_b - x_a)(y_b - y_a)|) + E_{p^*}[U(d(p^{**},p^*))]
\end{aligned}
$$

Where L is compact and $EU^{farmer}(a,b)$ continuous.

Question 4

Which fixed point theorem would you use to show the following? Use the relevant theorem to prove the result

Part 1.- Let p be a vector of non-negative prices in $\mathbb{R}^n$. If firms expect price vector p then they will produce a vector of outputs $S(p^*) \in \mathbb{R}^n$, where S is continuous function. For a vector of outputs, then prices will be determined by $D^{-1}(q)$, where D^{-1} is the continuous inverse demand function. There is a rational expectations equilibrium of this system (assume that supply and demand are homogeneous degree 1)

Given that the supply and demand are homogenous of degree 1 it make sense to normalized prices such that $\sum p_i = 1$. Therefore, we are going to work in a compact set of $\mathbb{R}^n$. We are looing for a rational expectations equilibrium of the form $p^* = D^{-1}(S(p^*))$. We know that D^{-1} it is well define and continuous and S is a continuous function, so we know that $f(x) = D^{-1}(S(x))$ is a well defined continuous function (why?). Then we are in the conditions of Brouwer Fixed Point Theorem (why?), and therefore we know that there exists a $p^* \in \{p \in \mathbb{R}^n : \sum p_i = 1\}$ such that $p^* = D^{-1}(S(p^*))$, that is p^* is a rational expectations equilibrium.

Part 2.- Any game with 2 players and 2 strategies for each player has a nash equilibrium in mixed stategies

Let assume that we have to players, I and II, each of the has two possible strategies: a_I and b_I, and a_{II} and b_{II} respectively. We are looking for a Nash equilibrium in mixed strategies, that is p^* and q^*, such that $p^* = p^*(q^*(p^*))$, that is the equilibrium strategy profile for each player is the best response to the profile chosen by the other player.

We can prove that we are in the conditions of Kakutani´s FPT, and therefore we know that there exists an equilibrium. That is let´s consider the best reaction function for both individuals

$p^*(q) = \arg\max_{p\in[0,1]} p[qu_I(a_I,a_{II}) + (1-q)u_I(a_I,b_{II})] + (1-p)u[qu_I(b_I,a_{II}) + (1-q)u_I(b_I,b_{II})]$ and

$q^*(p) = \arg\max_{q\in[0,1]} q[pu_{II}(a_{II},a_I) + (1-p)u_{II}(a_{II},b_I)] + (1-q)u[pu_{II}(b_{II},a_I) + (1-p)u_{II}(b_{II},b_I)]$

[you can worker a little more to get an expression of the best response functions, but this was enough to

Then consider the cartesian product of the two, that is $b(p,q) = p^*(q) \times q^*(p)$, where $b(p,q)$ is a self map that maps from a convex, non-empty and compact set to itself (why?), that has a closed graph, therefore we can apply Kakutani´s FPT.

[you can prove that every finite strategice-form game has a mixed strategy NE using the same argument