Midterm Solutions Econ 2010 [Fall 2014]

December 6, 2014

Question 1

Example 1. Let $f : M \to Y$ with $M = (0, 1)$ be given by $f(x) = x$. The set $A = (0, 1)$ is closed and bounded on M (but it is not compact) and the image of that set $f(A) = (0, 1)$ which is not compact. This function does not attain a maximum in that set.

Question 2:

Consider the following problem: You are trying to decide where to sit in the bar. There are currently two people in the bar, one of which you really like (l) and one of which you really dislike (d). Your happiness is going to be an increasing function of distance from the person you like and an increasing person of the distance from te person you dislike.

1. Write this problem formally. Give conditions on the elements of the problem to guarantee it to have a solution.

Normalize the area of the bar to the interval $[0, 1]^2$

A utility funciton $u : [0, 1]^2 \times [0, 1]^2 \times [0, 1]^2 \mapsto \mathbb{R}, u(x_l, x_d, x) = u(d(x, x_l), d(x, x_d))$ with $u(\cdot, d(x, x_d))$ increasing in its argument and $u(d(x, x_l), \cdot)$ decreasing in its argument.

The problem is to maximize

$max_{x \in [0,1]^2} u(x_l, x_d, x)$

Assume u is continuous and by Weirestrass we conclude that the problem has a solution.

2. First we prove that $P(x_l, x_d)$ is compact valued. Assuming that B is compact as in (1), take a sequence $\{y_i\}$ in $P(x_l, x_d)$ that converges to a point $y \in B$. We know, that it must be the case that $u(y_i) = u^* = V(x_l, x_d)$ for all y_i.

This implies that $u(y_i) \to y^*$ and by continuity of u, $u(y) = u^*$ and this in turn implies that $y \in P(x_l, x_d)$.

Now we use the sequential characterization of upper hemicontinuity. Let $\{x_l^i, x_d^i\}_i$ be a sequence in B that converges to (x_l, x_d), again let $\{y^i\}$ be a sequence in $P(x_l, x_d)$. This is a sequence in B that by (1) has a convergent subsequence in B with a limit point y. Now say the convergent subsequence $\{y^k\}$ and the associated sequence of parameters $\{x_l^i, x_d^i\}$ be the associated sequence of parameters.

Let $\overline{u}(y, x_l, x_d) = u(d(y, x_l), d(y, x_d))$ and note that if u is continuous.

Assume that

$$u(y^i, x_l^i, x_d^i) \geq u(y, x_l^i, x_d^i)$$

for all i, $y \in B$ but

$$u(y, x_l, x_d) < u(y^*, x_l, x_d)$$

for some $y \neq y^* \in B$.

By the continuity of u, $limu(y^*, x_l^i, x_d^i) = u(y^*, x_l, x_d) > u(y, x_l, x_d) = limu(y^i, x_l^i, x_d^i)$.

But this implies that there exists some N such that,

$n > N$

$u(y^*, x_l^i, x_d^i) > u(y^i, x_l^i, x_d^i)$.

This is a contradiction so $P(x_l, x_d)$ is closed.

3. Give an example of this problem in which a soultion is guaranteed to exists for any x_l and x_d, but P is not continuous.

$u(x_l, x_d, x) =$

$u(x_l, x_d, x) = -d(x_l - x) + d(x_d - x)$

subject to

$\Gamma(x_l, x_d) = bndry(B(0, 1))$ the boundary of the closed unit ball in $\mathbb{R}^2$.

In this case by the theorem of the maximum we are sure $P(x_l, x_d)$ is UHC.

We just have to prove it is not lower hemi-continuous. When

x_l and x_d are sitting together then there are two solutions at oposite sides of the circunference. Since the solution consits in two separated singletons, whenever they are not together there is only one solution so take a sequence

$x_l^n = \overline{x}_d + \frac{1}{n}$ then $(x_l^n, x_d) \to (\overline{x}_d, \overline{x}_d)$ but $\Gamma(\overline{x}_d, \overline{x}_d) = \{x^*\} \cup \{y^*\}$

at the same time $\Gamma(x_l^n, x_d) = \{z^{n,*}\}$ different from x^* and y^* and $z^{n,^*} \to z^*$ different from either x^* or y^* so there is a pair in the graph of Γ (x_l, x_d, y^*) or (x_l, x_d, x^*) such that no sequence of $(x_l, x_d, z^{*,n})$ converges to it.

4. Given an example of the problem in which a solution is guaranteed to exist, P is upper hemicontinuous, but V is not continuous.

$max_{x \in B} u(x_l, x_d, x) = -|x_l - x| + 2|x_d - x|$

and let $B = \{a\}$ if $x_l = x_d$ the barman puts all three acquaintances in a private room.

$B = [b, c]$ if $x_l \neq x_d$ they go to the main room in the bar if they do not sit together.

For $a < b < c$ then the objective is discontinuous. Since, $x_l^n = \overline{x}_d + \frac{1}{n}$ and x^{*n} is the solution when $x_l \neq x_d$ the objective is

$lim_{n \to \infty} - 2|\overline{x}_d - x^* + \frac{1}{n}| + |x_d - x^*| > 0$ but $x_l = x_d$ then $-|x_l - y^*| + 2|x_d - y^*| = 0$ when $y^* = x_l = x_d$.

5. Now consider the problem of the barman who can choose where l and d sit in the bar. The barman wants to maximize his or her revenue. The happier

each consumer is, the longer they will stay and the more they spend. l and d don't care where they are in relation to other people, but both have a favorite spot. Their happiness depends on their distance from that spot. You (as in the problem 1) get to choose your position in the bar having seen the location of l and d. Provide conditions on the primitive of the problem that ensure that the barman's problem has a solution.

I will assume this is a two stages problem, and assume common knowledge.

Keep the conditions in (1) so that $P(x_l, x_d)$ exists is uppper hemicontinuos and is compact valued and $V(x_l, x_d)$ is continuous.

The barman takes this infomation as given and maximizes her profit

$\pi(x_l, x_d, V(x_l, x_d)) = d(x_l, x_l^*) + d(x_d, x_d^*) + V(x_l, x_d)$ when the metrics are continuous then π is continuous since V is continuous.

Under the assumptions in (1) B is compact so the problem

$max_{(x_l,x_d)\in B}\pi(x_l, x_d, V(x_l, x_d))$

has a solution.

Question 3.

We say a collection of sets $\mathcal{A}$ of some space X has the finite intersection property if any finite collection of sets $\mathcal{A}$ have a non-empty intersection.

1. Let $X = (0, 1]$

$\mathcal{A}_n = (0, \frac{1}{n}]$ for all $n \in \mathbb{N}$

$\cap_{i=1}^{K} A_i \neq \emptyset$

$\cap_{i\in\mathbb{N}} A_i = \emptyset$

2. Let X be compact, now we have to show that $\cap A$ can be non empty.

Let X finite, then finite intersection is the same as arbitrary intersection.

Note: The intended wording of the question for part (2) was

2') Show that if X is compact, then if a collection of closed subsets $\mathcal{A}$ has the finite intersection property it must be the case that $\cup\mathcal{A}$ is non-empty.

Assume that X is compact, and take a collection $\mathcal{A}$ with the finite interesection property such that $\cup\mathcal{A} = \emptyset$, then take $X = \cap\mathcal{A}^c$. Also by the finite intersection property $\cap_n\mathcal{A} = Y \subseteq X$ and by complementation $\cup_n\mathcal{A}^c = X\backslash Y \subset X$ (because $Y \neq \emptyset$) so this says that there is no finite open cover of X which implies that this is not compact. This is a contradiction.

1 Question 4.

V is a lienar space and let L and K be linear operators on V. Show that $LK(x) = L(x)K(x)$ is a linear operator if and only if $LK(x) = 0$ for all $x \in V$.

By linearity

$LK(\lambda x) = L(\lambda x)K(\lambda x)$ by definition and by linearity of K, L

$LK(\lambda x) = \lambda^2 L(x)K(x)$ by associativity.

$\lambda^2 L(x)K(x) = \lambda L(x)K(x)$ so this holds in general when $L(x) = 0$ and $K(x) = 0$.

If $LK(x) = 0$ for all x then
$LK(x + y) = 0$ and because $LK(x) + LK(y) = 0$ it fulfills the first property
$LK(\lambda x) = \lambda LK(x) = 0$ for all $\lambda, x \in V$ so LK is linear.