Real Analysis on Metric Spaces

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1 Lecture 1

The first topic that we are going to cover in detail is what we'll call 'real analysis'. The foundation for this bit of the course is the definition of a 'metric', which is basically a way of measuring the distance between objects. This is something that was a property missing from our discussion of ordered fields in week 1 There is a very natural way of defining a metric on $\mathbb{R}^n$, but as we shall see the concept is much more general than that. We will use the concept of a metric to generate a *topology* on the spaces we are interested - or a description of what is open and closed. By doing so, we will be able to discuss various topics such as continuity, connectedness, compactness and so on. These in turn will allow us to prove results of more direct relevance to us as economists - such as Weierstrass theorem (which tells us when an optimization problem is guaranteed to have a solution), the Theorem of the Maximum, (which tells us how the solution to a maximization problem will change with the parameters of that problem), and some fixed point theorems (which are used in proving that equilibria of certain systems exist).

1.1 Definition of a Metric

To begin with we need to define a **metric**.

Definition 1 *A metric space (M, d) is a set M and metric $d : M \times M \to \mathbb{R}$ such that the following properties hold*

1. $0 \leq d(x, y) < +\infty \ \forall \ x, y \in M$

2. $d(x, y) = 0$ *if and only if* $x = y$

3. $d(x, y) = d(y, x)$

4. $d(x, y) \leq d(x, z) + d(z, y) \ \forall \ x, y, z \in M$ *(the triangle inequality)*

As usual, there is a very natural way of defining a metric in $\mathbb{R}^n$, which is the Euclidian Metric

Definition 2 *The Euclidian metric is a metric on* $\mathbb{R}^n$ *such that*

$$d(x, y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{\frac{1}{2}}$$

For $\mathbb{R}$, this reduces to $d(x, y) = |x - y|$, which clearly satisfies the necessary properties. We can demonstrate easily in $\mathbb{R}^2$ that the Euclidian metric is indeed a metric. To prove this formally for $\mathbb{R}^n$, the only tricky bit (check) is the triangle inequality. But for this we can use a result called Minkowski's inequality 1. This this states that

Theorem 1 *(Minkowski's Inequality 1): For any* $a, b \in \mathbb{R}^n$ *and* $1 \leq p < \infty$

$$\left(\sum_{i=1}^{n} |a_i + b_i|^p \right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n} |a_i|^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} |b_i|^p \right)^{\frac{1}{p}}$$

Proof. *This is trivial for the case where* $a = \emptyset$ *or* $b = \emptyset$, *so focus on the case where* $\alpha = (\sum_{i=1}^{n} |a_i|^p)^{\frac{1}{p}}$ *and* $\beta = (\sum_{i=1}^{n} |b_i|^p)^{\frac{1}{p}}$ *are positive real numbers. Define the vectors* $\bar{a} = \frac{1}{\alpha} a$ *and* $\bar{b} = \frac{1}{\beta} b$. *This implies that*

$$\begin{aligned}
& \sum_{i=1}^{n} |\bar{a}_i|^p \\
= & \sum_{i=1}^{n} |\frac{1}{\alpha} a_i|^p \\
= & \sum_{i=1}^{n} |\frac{1}{(\sum_{i=1}^{n} |a_i|^p)^{\frac{1}{p}}} a_i|^p \\
= & \ 1 \\
= & \sum_{i=1}^{n} |\bar{b}_i|^p
\end{aligned}$$

As the triangle inequality holds for the absolute value function, and that raising a positive number to the power p is an increasing function, we get

$$\begin{aligned} |a_i + b_i|^p &\leq (|a_i| + |b_i|)^p \\ &= (\alpha|\bar{a}_i| + \beta|\bar{b}_i|)^p \\ &= (\alpha+\beta)^p \left(\frac{\alpha}{(\alpha+\beta)}|\bar{a}_i| + \frac{\beta}{(\alpha+\beta)}|\bar{b}_i|\right)^p \end{aligned}$$

As t^p *is a convex function on* $\mathbb{R}_+$*, we get*

$$\left(\frac{\alpha}{(\alpha+\beta)}|\bar{a}_i| + \frac{\beta}{(\alpha+\beta)}|\bar{b}_i|\right)^p \leq \frac{\alpha}{(\alpha+\beta)}|\bar{a}_i|^p + \frac{\beta}{(\alpha+\beta)}|\bar{b}_i|^p$$

so

$$|a_i + b_i|^p \leq (\alpha+\beta)^p \left(\frac{\alpha}{(\alpha+\beta)}|\bar{a}_i|^p + \frac{\beta}{(\alpha+\beta)}|\bar{b}_i|^p\right)$$

Summing over i

$$\begin{aligned} \sum_{i=1}^{n} |a_i + b_i|^p &\leq (\alpha+\beta)^p \left(\frac{\alpha}{(\alpha+\beta)}\sum_{i=1}^{n}|\bar{a}_i|^p + \frac{\beta}{(\alpha+\beta)}\sum_{i=1}^{n}|\bar{b}_i|^p\right) \\ &= (\alpha+\beta)^p \left(\frac{\alpha}{(\alpha+\beta)} + \frac{\beta}{(\alpha+\beta)}\right) \\ &= (\alpha+\beta)^p \end{aligned}$$

raising both sides to the power $\frac{1}{p}$ *gives*

$$\begin{aligned} \left(\sum_{i=1}^{n} |a_i + b_i|^p\right)^{\frac{1}{p}} &\leq (\alpha+\beta) \\ &= \left(\sum_{i=1}^{n}|a_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n}|b_i|^p\right)^{\frac{1}{p}} \end{aligned}$$

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Thus, letting $\rho = 2$ and $a = x - z$ and $b = z - y$ we get

$$\begin{aligned} \left(\sum_{i=1}^{n} ((x_i - z_i) + (z_i - y_i))^2\right)^{\frac{1}{2}} &\leq \left(\sum_{i=1}^{n} ((x_i - z_i))^2\right)^{\frac{1}{2}} + \left(\sum_{i=1}^{n} ((z_i - y)_i)^2\right)^{\frac{1}{2}} \\ \left(\sum_{i=1}^{n} (x_i - z_i)^2\right)^{\frac{1}{2}} &\leq \left(\sum_{i=1}^{n} (x_i - z_i)^2\right)^{\frac{1}{2}} + \left(\sum_{i=1}^{n} (z_i - y_i)^2\right)^{\frac{1}{2}} \\ &\Rightarrow d(x,y) \leq d(x,z) + d(z,y) \end{aligned}$$

As usual, there are other examples of metrics. In fact, Minkowsiki tells us that, for any ρ such that $1 \leq \rho < \infty$,

$$d_\rho(x,y) = \left(\sum_{i=1}^{n} (|x_i - y_i|)^\rho \right)^{\frac{1}{\rho}}$$

is a metric. However, we can also define metrics in all sorts of weird and wonderful ways

Example 1 *The discrete metric. Let M be any non-empty set and define $d(x,y)$ as*

$$\begin{aligned} d(x,y) &= 0 \text{ if } x = y \\ &= 1 \text{ otherwise} \end{aligned}$$

then M, d form a metric space.

Proof. *The only non-trivial bit is the triangle inequality, but this is also obvious. If $x = y$ then it holds trivially. If $x \neq y$ then $z = y$ and $z = x$ cannot simultaneously be true, so $d(x,z) + d(z,y) \geq 1 = d(x,y)$ and we are done.* ■

Example 2 *Let T be an non-empty set and $\mathcal{B}(T)$ be the set of all bounded real functions. Define the sup-metric for $f, g \in \mathcal{B}(T)$ as*

$$d_\infty(f,g) = \sup\left[|f(x) - g(x)| : x \in T\right]$$

then $(\mathcal{B}(T), d_\infty(f,g))$ is a metric space.

Proof. *Homework* ■

1.2 Defining a Topology

We are now going to use the concept of a metric to define a **topology** on our metric space. A topology is just a description of what defines a subset as **open**

Definition 3 *Let X be an arbitrary set. A topology on X is a collection $T \subset 2^X$ such that*

1. *The empty set and $X \in T$*
2. *if $\{T_\alpha\}$ is a collection of sets in T then $\cup_\alpha T_\alpha \in T$*

3. if $\{T_\alpha\}_{i=1}^n$ is a finite collection of sets in T then $\cap_\alpha T_\alpha \in T$

In other words, a topology on any set X is any collection of subsets that contains the empty set and X and is closed under union and finite intersection. We will call sets in T the open sets as defined by that topology.

Any metric can generate a topology. To see this, we need some more definitions

Definition 4 *Let M, d be a metric space and $r \in \mathbb{R}_{++}$. The set $B(x, r)$, defined as*

$$B(x, r) = \{y \in M | d(x, y) < r\}$$

is called an open ball, with a centre x and radius r. The set $\bar{B}(x, r)$ defined as

$$\bar{B}(x, r) = \{y \in M | d(x, y) \leq r\}$$

is called a closed ball

Note here that we are sneakily using an implicit topology on $\mathbb{R}$, by which open intervals are open.

Now, we can use our metric to define a topology

Definition 5 *Let M, d be a metric space. Then a set $S \subset M$ is open, if, for every $x \in S$ there exists an r such that $B(x, r) \subset S$*

Given this claim, we want to check a few things. First, we want to check that an open ball is itself an open set

Lemma 1 *An open ball is an open set*

Proof. *Let $B(x, r)$ be some open ball, and let $y \in B(x, r)$. Let $s = \frac{1}{2}(r - d(x, y))$. By definition of the open ball, we know that $s > 0$, so we can define the open ball $B(y, s)$. Our claim is that $B(y, s) \subset B(x, r)$. To see this, pick $z \in B(y, s)$, and note that*

$$\begin{aligned} d(z, x) &\leq d(z, y) + d(y, x) \\ &< s + d(y, x) \\ &= \frac{1}{2}r + \frac{1}{2}d(x, y) \end{aligned}$$

but, as $\frac{1}{2}d(x,y) < \frac{1}{2}r$

$$\frac{1}{2}r + \frac{1}{2}d(x,y) < r$$

and so $z \in B(x,r)$ ■

We are going to be using arguments such as this an awful lot over the next few lectures.

Next, I need to come good on my claim that our definition of open sets is a topology

Theorem 2 *The collection* $T = \{S \subset M | \ \forall \ x \in S \ \exists \ r \in \mathbb{R}_{++}$ *such that* $B(x,r) \subset S\}$ *is a topology*

Proof. *This requires us to prove three things*

1. *The empty set and* $X \in T$*. This is trivial*
2. *If* $\{T_\alpha\}$ *is a collection of sets in* T *then* $\cup_\alpha T_\alpha \in T$*. Let* $x \in \cup_\alpha T_\alpha$*. Then* $\exists \ T_\alpha$ *such that* $x \in T_a$*. As* T_a *is open, there exists some* r *such that* $B(x,r) \subset T_a$*. But then* $B(x,r) \subset \cup_\alpha T_\alpha$
3. *if* $\{T_\alpha\}_{i=1}^n$ *is a finite collection of sets in* T *then* $\cap_\alpha T_\alpha \in T$*. Let* $x \in \cap_\alpha T_\alpha$*. Then for every* $\alpha \in \{1,,,n\}$ *there exists an* $r(\alpha)$ *such that* $B(x, r(\alpha)) \subset T_\alpha$*. Let* $r = \min_\alpha r(\alpha)$*. (Note how we have used the finiteness of* $\{T_\alpha\}_{i=1}^n$*). Then, for every* α*,* $B(x,r) \subset B(x,r(\alpha)) \subset T_\alpha$*, and so* $B(x,r) \subset \cap_\alpha T_\alpha$

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You might be wondering about the asymmetry between the unrestrictedness of the unions and the fineniteness of the intersections. Let me show you that this is a necessary restriction, at least if we want our metric-based definition to work as a topology

Remark 1 *The intersection of a countable collection of open sets* $\{T_\alpha\}_{i=1}^\infty$ *is not necessarily open. To see this, you should prove to yourself that open intervals on the real line are open. Then define* $T_\alpha = (0, 1 + \frac{1}{\alpha})$ *for* $\{T_\alpha\}_{\alpha=1}^\infty$*. Then note that*

$$\cap_\alpha T_\alpha = (0,1]$$

as, clearly $1 \in (0, 1 + \frac{1}{\alpha}) \ \forall \ \alpha$*, but for any* $r > 1$*, we can find some* α *such that* $1 + r \notin T_\alpha$*. But* $(0,1]$ *is not open, as for any* $r > 0$*,* $B(1,r) \not\subset (0,1]$

Note that, for any given space (say $\mathbb{R}^n$) different metrics can give rise to different topologies. For example - what sets are open in $\mathbb{R}^n$ under the discrete metric? However, in some cases, different metrics will give rise to the same topologies. For example, consider the family of metrics d_ρ we defined above. It should be intuitively obvious that any two metrics in this family will define the same topology on $\mathbb{R}^n$. Clearly these metrics are 'equivalent' in some way. This is a useful fact, as certain properties may be easier to prove in (say) d_1 than in d_2. The equivalence of these metrics means that proving things in d_1 may be enough to guarantee that they hold for other metrics.