3 Lecture 3

3.1 Continuity

Next, we move onto the idea of continuity - a property of functions that I am sure you have all used informally, and we will now define formally. This is an incredibly useful property when it comes to optimization, as we will soon see. Broadly speaking, a continuous function is one that does not jump.

Definition 10 *Let (X, d) and (Y, ρ) be 2 metric spaces. A function $f : X \to Y$ is continuous at a point $x \in X$ if, for every $\varepsilon > 0$, $\exists\ \delta > 0$ such that*

$$\begin{aligned} d(x, x') &< \delta \\ &\Rightarrow \rho(f(x), f(x')) < \varepsilon \end{aligned}$$

Equivalently, $f(B(x, \delta)) \subset B(f(x), \varepsilon)$. We say a function is continuous if it is continuous at every point in its domain.

For a real valued function endowed with the standard metric, it should be pretty easy to see that this definition is equivalent to our intuition that a continuous function is one that can be drawn without the pen leaving the paper.

Note that whether or not a particular function is continuous depends crucially on the metric in use, as the follow example shows

Example 6 *Any function defined on $f : Y \to X$ is continuous if Y is endowed with the discrete metric. To see this, note that the ball $B(x, \frac{1}{2}) = \{x\}$ for all $x \in Y$, thus, for any $\varepsilon > 0$, $f(B(x, \frac{1}{2})) \subset B(f(x), \varepsilon)$. Thus, if we consider the indicator function $f : Y \to \mathbb{R} = 1_0$, then this function is discontinuous if $Y = \mathbb{R}$ endowed with the standard metric, but is continuous if Y is $\mathbb{R}$ endowed with the discrete metric.*

There are several alterative characterizations of continuity that are equivalent

Theorem 3 *Let $f : (X, d) \to (Y, \rho)$. The following are equivalent*

1. *f is continuous*

2. *for any $x \in X$ and sequence $x_n \to x$ implies $f(x_n) \to f(x)$*

3. *$f^{-1}(S)$ is open for any open set S*

4. *$f^{-1}(S)$ is closed for any closed set S*

Proof. *We do this in four parts*

1. *(1 implies 2). Take any ε. We need to find an N such that $n > N \Rightarrow \rho(f(x_n), f(x)) < \varepsilon$. By continuity, we know that there must be a δ such that $d(x, x') < \delta$ implies that $\rho(f(x), f(x')) < \varepsilon$. Moreover, as $x_n \to x$, we can find some N such that $d(x_n, x) \leq \delta \ \forall \ n > N$. Thus, for $\forall$ $n > N$, $\rho(f(x_n), f(x)) < \varepsilon$*

2. *(2 implies 4) Let $S \subset Y$, closed. Pick any convergent sequence $x_n \to x$ in $f^{-1}(S)$. We need to show that $x \in f^{-1}(S)$. By the line above, we know that $f(x_n) \to f(x)$. But as $f(x_n)$ is a sequence in S, which is closed, $f(x) \in S$ and so $x \in f^{-1}(S)$*

3. *(4 implies 3).Let $S \subset Y$, open. By definition, Y/S closed. Thus, by the line above, $f^{-1}(Y/S)$ closed. Thus, $X/f^{-1}(Y/S)$ open. But this is just $f^{-1}(S)$*

4. *(3 implies 1). Pick any $x \in X$ and $\varepsilon > 0$. As $B(f(x), \varepsilon)$ is open, then so is $f^{-1}(B(f(x), \varepsilon)$. Note that $x \in f^{-1}(B(f(x), \varepsilon)$. By the definition of an open set $\exists$ δ such that $B(x, \delta) \subset f^{-1}(B(f(x), \varepsilon)$. Thus, $f(B(x, \delta)) \subset B(f(x), \varepsilon)$.*

■

It should be obvious that it is not necessarily true that a continuous function will map an open set to an open set. You will be asked to come up with a counterexample for your homework

We will now go through some more properties of continuous functions which show that various combinations of continuous functions also tend to be continuous.

Lemma 4 *Let X, Y and Z be metric spaces, and $f : X \to Y$, $h : X \to Y$ and $g : Y \to Z$ be continuous functions.*

1. *$g \circ f : X \to Z$ is continuous.*

2. *$j : X \to Y^2$ such that $j(x, x) = (f(x), h(x))$ is continuous, where Y^2 is endowed with the product metric*[2]

3. *$f \pm g$, $f.g$, $\max(f, g)$ and $\min(f, g)$ are all continuous*

Proof. *Exercise* ■

Before moving on, it is worth considering some stronger forms of continuity

Definition 11 *Let X and Y be two metric spaces. A function $f : X \to Y$ is uniformly continuous if, for every ε, there exists a δ such that $f(B(x, \delta)) \subset B(f(x), \varepsilon)\ \forall\ x \in X$*

Spot the difference between uniform continuity and standard continuity? For standard continuity, for any ε, we are allowed to find a δ *for each* x such that $f(B(x, \delta)) \subset B(f(x), \varepsilon)$. For uniform continuity you have to pick the *same* δ for *every* x. Clearly, any uniformly continuous function is continuous, but, the reverse is not true.

When does this make a difference? A classic example is the function $f : \mathbb{R}_{++} \to \mathbb{R}$ $f(x) = \frac{1}{x}$. This function is clearly continuous (you should check), but it is not uniformly continuous, select $\varepsilon = 1$. If the function is uniformly continuous, then we must be able to find a δ such that $f(B(x, \delta)) \subset B(f(x), 1)\ \forall\ x$. Say we find such a δ. It has to be the case that $|\frac{1}{x} - \frac{1}{y}| < 1$ for all x and y such that $|y - x| < \delta$. In other words, $|x - y| < xy$. But set $y = x + \frac{\delta}{2}$. Then $|\frac{\delta}{2}| < x^2 + x\frac{\delta}{2}$, for all $x > 0$. This is clearly not true

A further strengthening of the concept of continuty is *Lipschitz Continuity*

Definition 12 *Let X and Y be two metric spaces. A function $f : X \to Y$ is Lipschitz continuous if there exists a real number $K \in \mathbb{R}_+$ such that*

$$d(f(x), f(y)) \leq K d(x, y)$$

[2]For a collection $\{X_i, d_i\}_{i=1}^N$ of metric spaces, the product metric on $X = \times^N X_i$ is given by

$$d(x, y) = \sum_{i=1}^{N} d(x_i, y_i)$$

Notice that, for real valued functions, this implies that the slope of the line between any two points x and y must be bounded above by K. The standard example of a function that is uniformly but not Lipschitz continuous if $y^{\frac{1}{2}}$