6 Lecture 6

6.1 Continuity of Correspondances

So far we have dealt only with functions. It is going to be useful at a later stage to start thinking about correspondances. A correspondance is just a set-valued function: a correspondance from X to Y is a map that takes every element in X and maps it to a non-empty subset of Y (note that a correspondance is therefore also a function if we define the range correctly, yes?). If Γ is a correspondance from X to Y we write $\Gamma : X \rightrightarrows Y$.

Note that we have (formally or informally) come across a number of correspondances in economics. For example, in a world with n commodities and a fixed income level I, we can think of a budget set as a correspondance $B : \mathbb{R}^n_+ \rightrightarrows \mathbb{R}^n_+$ defined as

$$B(p) = \left\{x \in \mathbb{R}^n_+ | px \leq I\right\}$$

Similarly, for any set X endowed with a preference relation $\succeq$, we can think of the upper contour set as a correspondance $U_{\succ} : X \rightrightarrows X$ defined as

$$U_{\succ}(x) = \{y \in X | x \succ y\}$$

As with functions, it is going to be useful to think of the concept of continuity with regard to correspondances:

Definition 26 *A correspondance $\Gamma : X \rightrightarrows Y$ is upper-hemicontinuous at $x \in X$ if for every open subset $O \subset Y$ with $\Gamma(x) \subseteq O$ there exists a $\delta > 0$ such that*

$$\Gamma(B(x, \delta)) \subseteq O$$

It is lower-hemi-continuous at x if, for every open set O in Y such that $\Gamma(x) \cap O \neq \emptyset$ then there exists a δ such that

$$\Gamma(x') \cap O \neq \emptyset \;\forall\; x' \in B(x, \delta)$$

It is continuous if it is both upper and lower hemicontinuous

We will draw graphs to demonstrate these properties in class.

As with continuous functions, there is a sequential characterization of both upper and lower hemicontinuity, that we will state but not prove:

Lemma 13 *A correspondance* $\Gamma : X \Rightarrow Y$ *is lower hemicontinuous at* $x \in X$ *if and only if, for any seqeunce* $x_m \to x$ *in* X*, and any* $y \in \Gamma(x)$*, there exists a sequence* $y_m \to y$ *such that* $y_m \in \Gamma(x_m)$ $\forall$ x_m

In order to state a similar result for upper-hemi continuity, we need to define the concept of a compact-valued correspondance

Definition 27 *A correspondance* $\Gamma : X \Rightarrow Y$ *is compact valued if, for every* $x \in X$, $\Gamma(x)$ *is compact*

The concepts of closed-valued and convex-valued are defined analogously

So, what about UHC correspondances?

Lemma 14 *Let* $\Gamma : X \Rightarrow Y$ *be a correspondance. If, for every* $x_m \to x$ *in* X*, and* $y_m \in \Gamma(x_m)$ *there exists a subsequence of* y_m *that converges to a point in* $\Gamma(x)$*, then* Γ *is upper hemi continuous. If* Γ *is also compact valued, then the converse is also true.*

One other useful property of UHC and compact valued correspondances is the following:

Proposition 4 *Let* $\Gamma : X \Rightarrow Y$ *be an upper hemi-continuous and compact valued correspondance. Then* $\Gamma(S)$ *is compact in* Y *for any compact subset* S *of* X[5]

6.2 Applications

We are now going to make use of some of the machinery that we have developed in order to prove some genuinely useful results. In fact, another title for this section could have been 'Some Reasons

[5]Note that we define $\Gamma(S)$ as follows:

$$\Gamma(S) := \cup_{x \in S} \Gamma(x)$$

NOT as $\{\Gamma(x) | x \in S\}$. I.e. it is a subset of Y, not a collection of sets in Y.

Why we Care About the Rest of this Chapter.' We are basically going to show some things that are true about compact and complete sets that are going to be genuinely useful, even outside this course. Gasp! In particular, we will show the following

1. Any continuous real valued function obtains a maximum and minimum value when evaluated on a compact metric space (Weierstarss's theorem)
2. A certain class of functions is going to have a fixed point on a complete space (Banach Fixed Point Theorem)
3. The Theorem of the Maximum

So here we go

Theorem 11 (Weierstrass) *Let X be a compact metric space and $f : X \to \mathbb{R}$ be continuous, then f attains its max and min in X*

Proof. *This theorem states that there exists $x^* \in X$ such that $f(x^*) = \sup_{x \in X} f(x) = \max_{x \in X} f(x)$, and the same for the minimim. We will prove it for the maximum - an equivalent method will work for the minimum.*

By theorem 10 we know that $f(X)$ is compact, and so (as $f(X) \subset \mathbb{R}$) closed and bounded. But this means that $\sup_{x \in X} f(x) < +\infty$. Also, as $\sup_{x \in X} f(x)$ is a closure point of $f(X)$, then $\sup_{x \in X} f(x) \in f(X)$. This implies their exists some x^* such that $(x^*) = \sup_{x \in X} f(x) = \max_{x \in X} f(x)$ ■

Given the machinery that we have built, this is a very simple result, but one that is very useful - it gives you a condition under which optimization problems will actually have solutions!

Next we are going to move on to Banach Fixed Point theorem. In general, fixed point theorems are very useful classes of result that give us conditions under which for some function $f : X \to X$ we can find a value such that $f(x) = x$. These results are incredibly useful when it comes to proving the existence of various types of equilibria. There are lots of different fixed point theorems, that provide different conditions under which fixed points exist. We will hopefully get to some others later in the course.

In order to state Banach, we are going to have to introduce some preliminaries.

Definition 28 *Let X be a metric space, and $f : X \to X$. We will say that f is a contraction if there exists some $0 < k < 1$ such that*

$$d(f(x), f(y)) \leq kd(x, y) \; \forall \; x, y \in X$$

The inf of such k's is called the contraction coefficient

So a contraction is a function that maps X to itself (also called a **self map**) such that the function spits out items that are closer together than what you put into it. The most obvious contraction is the function $f : \mathbb{R} \to \mathbb{R}$ such that $f(t) = \alpha t$ for $-1 < \alpha < 1$

Why do we care about contractions? The reason is, because of Banach, we know that contractions on complete metric spaces have a fixed point, and as I have already discussed, fixed points are nice things.

Theorem 12 (Banach Fixed Point Theorem) *Let X be a complete metric space, and f be a contraction on X. Then there exists a unique x^* such that $f(x^*) = x^*$*

Proof. *We first show the existence of some $x^* \in X$ such that $f(x^*) = x^*$. Pick some $x^0 \in X$ and define a sequence recursively such that $x^{n+1} = f(x^n)$. The sequence $\{x_n\}_{n=1}^{\infty}$ is a Cauchy. To see this, let k be the contraction coefficient of f, and note that*

$$\begin{aligned} d(x^2, x^1) &= d(f(x^1), f(x^0)) \leq kd(x^1, x^0) \\ d(x^3, x^2) &= d(f(x^2), f(x^1)) \leq kd(x^2, x^1) \leq k^2 d(x^1, x^0) \end{aligned}$$

More generally, $d(x^{n+1}, x^n) \leq k^n d(x^1, x^0)$. Thus, for any l, $m > l + 1$ we have

$$\begin{aligned} d(x^m, x^l) &\leq d(x^m, x^{m-1}) + d(x^{m-1}, x^{m-2}) + .. + d(x^{l+1}, x^l) \\ &\leq (k^{m-1} + k^{m-2} + ... + k^l) d(x^1, x^0) \\ &= k^l \frac{(1 - k^{m-l})}{1 - k} d(x^1, x^0) \end{aligned}$$

so $d(x^m, x^l) < \frac{k^l}{1-k} d(x^1, x^0)$ implying that the sequence is cauchy.

As X is complete, it must be the case that $\{x_n\}_{n=1}^{\infty}$ converges to some point $x^* \in X$. Therefore, *for any* $\varepsilon > 0$*, there exists some* M *such that* $d(x^*, x^n) < \frac{\varepsilon}{2} \ \forall\ n > M$*. Thus*

$$\begin{aligned} d(f(x^*), x^*) &\leq d(f(x^*), x^{n+1}) + d(x^*, x^{n+1}) \\ &= d(f(x^*), f(x^n)) + d(x^*, x^{n+1}) \\ &\leq kd(x^*, x^n) + d(x^*, x^{n+1}) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

This is true for all $\varepsilon > 0$*, so it must be that* $d(f(x^*), x^*) = 0$*, and so* $f(x^*) = x^*$*. Thus,* x^* *is a fixed point.*

To prove uniqueness, note that, if $x \in X$ *was another fixed point of* f*, we would have* $d(x, x^*) = d(f(x), f(x^*))$*, a contradiction, as* $d(f(x), f(x^*)) < kd(x, x^*)$ *for some* $k < 1$*.* ■

7 Lecture 7

Finally we move on to the theorem of the maximum. This is going to be a very useful theorem, and it is designed to answer the following questions:

Example 13 *Let* $p \in \mathbb{R}^n_+$ *be a vector of prices,* I *be income, and consider an agent who choses bundles* $x \in \mathbb{R}^n_+$ *to maximize a utility function* $u : \mathbb{R}^n_+ \rightarrow \mathbb{R}$ *subject to the budget constraint*

$$B(p, I) = \left\{ x \in \mathbb{R}^n_+ | px \leq I \right\}$$

Let $D(p, I)$ *be the demand function, so that*

$$D(p, I) = \arg\max_{x \in \mathbb{R}^n} u(x) | px \leq I$$

And $v(p, I)$ *be the derived utility, so that*

$$v(p, I) = \max_{x \in \mathbb{R}^n} u(x) | px \leq I$$

Can we say anything about the properties of D *and* v*? In other words, do we know anything about how demand and derived utility change with the parameters of the problem?*

This is exactly what the theorem of the maximum tells us (under certain assumptions). In order to define these properties, we need to define the concept of the graph of a correspondance :

Definition 29 *The graph of a correspondance* $\Gamma : Y \Rightarrow X$ *is the set of pairs* $\{x, y\} \in X \times Y$ *such that* x *is in the correspondance evaluated at* y

$$Gr_\Gamma = \{\{x, y\} \in X \times Y | x \in \Gamma(y)\}$$

Theorem 13 (The Theorem of the Maximum) *Let*

- X *and* Y *be metric spaces (*Y *will be the set of things that are chosen,* X *the set of parameters)*
- $\Gamma : X \Rightarrow Y$ *be compact valued and continuous (this is the constraint set defined by the parameters)*

- *$f : X \times Y \to \mathbb{R}$ be continuous, (this is the utility function)*

  Now define $y^* : X \rightrightarrows Y$ as the set of maximizers of f given parameters x

$$y^*(x) = \arg\max_{y \in \Gamma(x)} f(x, y)$$

  and define $f^* : X \rightrightarrows Y$ as the maximized value of f for f given parameters x

$$f^*(x) = \max_{y \in \Gamma(x)} f(x, y)$$

Then

1. *y^* is upper hemi-continuous and compact valued*
2. *f^* is continuous*

Translating into the language of the example

- X is the set of price vectors and income
- Y is the commodity space
- Γ is the budget correspondance
- f is the utility function (note that we do not let utility depend directly on prices, but we could if we wanted to)
- y^* is the demand function
- f^* is the derived utility

This is a really cool result. With relatively few assumptions, we are able to guarantee some neat properties of things we really care about. The proof is somewhat cumbersome, so we will sketch it here.

Proof. We will prove this as a set of claims:

Claim 1: y^* has a closed graph. Let (y, x) be a closure point of Gr_{y^*}. We need to show that this is in Gr_{y^*}. First, we show that y is feasible at x, then we show that it maximizes f at x

Note that, if $(y.x)$ is a closure point of Gr_{y^*}, then we can construct a sequence $(y_n, x_n) \to (y, x)$ such that $(y_n, x_n) \in Gr_{y^*}$. This implies that $y_n \in y^*(x_n)\ \forall\ n$. This in turn implies that $y_n \in \Gamma(x_n)\ \forall\ n$. As Γ is UHC and compact valued, then y_n must have a subsequences that converges to some $y' \in \Gamma(x)$, but as $y_n \to y$, it must be that $y \in \Gamma(x)$, so y is feasible at x

Now assume that $y \notin y^*(x)$, then there must be some $\bar{y} \in \Gamma(x)$ such that $f(x, \bar{y}) > f(x, y)$. By LHC, there must be some sequence $\bar{y}_n \to \bar{y}$ such that $\bar{y}_n \in \Gamma(x_n)\ \forall\ n$. By the continuity of f, we know that

$$\begin{aligned} \lim f(x_n, \bar{y}_n) &= f(x, \bar{y}) \\ \lim f(x_n, y_n) &= f(x, y) \end{aligned}$$

But, as $y_n \in y^*(x_n)$, this implies that

$$\begin{aligned} f(x_n, y_n) &\geq f(x_n, \bar{y}_n)\ \forall\ n \\ f(x, y) &< f(x, \bar{y}) \end{aligned}$$

A contradiction (check)

Claim 2 y^* is UHC and compact valued. As $y^*(x)$ is closed (by the above result) and $y^*(x) \subset \Gamma(x)$ compact, it must be the case that $y^*(x)$ is compact, and so y^* is compact valued.. Let (x_n, y_n) be a sequence such that $x_n \to x$ and $y_n \in y^*(x_n) \subset \Gamma(x_n)\ \forall\ n$.By the UHC and compact valuedness of Γ, we know that there is a subsequence $\bar{y}_n$ that converges to some $y \in \Gamma(x)$. The closed graph property tells us that, as $(x_n, \bar{y}_n) \in Gr_{y^*}$, then $(x, y) \in Gr_{y^*}$, and so $y \in y^*(x)$, implying that y^* is UHC

Claim 3 f^* is continuous. Let $x_n \to x \in X$. We need to show that $f^*(x_n) \to f^*(x)$. We know that there is a subsequence $f^*(x_k) \to \limsup f^*(x_n)$. Pick a sequence $y_k \in y^*(x_k)$, so

$$f^*(x_k) = f(x_k, y^*(x_k)) = f(x_k, y_k)$$

Because y^* is compact valued and UHC, there is a subsequence $y_j \to y \in y^*(x)$. By the continuity of f, the fact that $x_j \to x$ and $y_j \to y$ implies that

$$\begin{aligned} & f^*(x_j) \\ = \ & f(x_j, y_j) \\ \to \ & f(x, y) \end{aligned}$$

. but as $y \in y^*(x)$, $f(x,y) = f^*(x)$, so $f^*(x)$ is the lim sup of $f^*(x_n)$. A similar argument proves that $f^*(x)$ is also the lim inf of $f^*(x_n)$, so we are done.

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