# Multi Variable Calculus

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## 1 Functions from $\mathbb{R}^n$ to $\mathbb{R}^m$

So far we have looked at functions that map one number to another number, i.e. that map from $\mathbb{R}$ to $\mathbb{R}$. Often, however, we encounter functions of more than one variable. For example, a utility function takes as input consumption of good 1, $x_1$ and consumption of good 2, $x_2$, and assigns to them a utility level: $u = u(x_1, x_2)$. In this case, the function maps from $\mathbb{R}^2$ ( $\mathbb{R}^2_+$ strictly speaking) to $\mathbb{R}$. In general, a function that takes $n$ input variables and maps them to one output variable is said to map from $\mathbb{R}^n$ to $\mathbb{R}$.

On the other hand, there are also functions that take one input variable and map it to several output variables. For example, a function that predicts the state of the economy takes time $t$ as input variable and assigns to it a level of GDP and of the capital stock: $E(t) = (Y(t), K(t))$. In this case, the function maps from $\mathbb{R}$ to $\mathbb{R}^2$. In general, a function that takes one input variables and maps it to $m$ output variables is said to map from $\mathbb{R}$ to $\mathbb{R}^m$. Note that a function $f : \mathbb{R} \to \mathbb{R}^m$ is really nothing else than an array of $m$ functions, each mapping from $\mathbb{R}$ to $\mathbb{R}$:

$$f(x) = (f_1(x), f_2(x), \ldots, f_m(x)).$$

Of course a function can take multiple input variables and multiple output variables. For example, a production function takes capital $K$, labor $L$ and intermediate product $x$ as input and assigns to it the level of final good 1, $y_1$, and final good, $y_2$: $(y_1, y_2) = f(K, L, x)$. In this case, the function maps from $\mathbb{R}^3$ to $\mathbb{R}^2$. In general, a function that takes $n$ input variables and maps them to $m$ output variables is said to map from $\mathbb{R}^n$ to $\mathbb{R}^m$. Note that again we can think of a function $f : \mathbb{R}^n \to \mathbb{R}^m$ as an array of $m$ functions, each mapping from $\mathbb{R}^n$ to $\mathbb{R}$:

$$f(x_1, x_2, \ldots, x_n) = (f_1(x_1, x_2, \ldots, x_n), f_2(x_1, x_2, \ldots, x_n), \ldots, f_m(x_1, x_2, \ldots, x_n)).$$

It is hard to visualize functions in higher dimensions graphically. One way to picture a function is to look at its level sets. Consider a function $f(\mathbf{x})$ with $\mathbf{x} = (x_1, \ldots, x_n)$. A **level set** is the set of all $\mathbf{x}$'s such that $f(\mathbf{x}) = c$, where $c$ is some specified constant. For example, indifference curves are level sets, as are isoquant curves.

## 2 Derivatives of a Single Function of Several Variables

### 2.1 Partial Derivatives

Let $f$ be a function of many variables, e.g. $f(x) = f(x_1, x_2, \ldots, x_n)$. The partial derivative of $f$ with respect to $x_1$, denoted $\frac{\partial f}{\partial x_1}$ or $f_{x_1}$, is the function obtained by differentiating $f$ with respect to $x_1$ and treating all other variables as constants. Rigorously, this is

$$f_{x_1}(x_1, x_2, \ldots, x_n) = \lim_{h \to 0} \frac{f(x_1 + h, x_2, \ldots, x_n) - f(x_1, x_2, \ldots, x_n)}{h}$$

**Interpretation:** $\frac{\partial}{\partial x_1} f(x)$ is the change in the value of the function if $x_1$ changes infinitesimally. It is the slope of the tangent line to the graph of $f$ at point $x$ in the $x_1$-direction.

**Example:**
Let $f(x, y) = x^2 y^3$. Then

$$\frac{\partial f}{\partial x} = 2xy^3$$

## 2.2 The Total Derivative or Gradient

The total derivative is merely a vector containing all of the partial derivatives of a function,

$$Df(x_1, x_2, \ldots, x_n) = \left( \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots \frac{\partial f}{\partial x_n} \right)$$

The transpose of this vector, evaluated at a specific point $a = (x_1^*, x_2^*, \ldots x_n^*)$ is called the gradient of $f$ at $a$, or $\nabla f(a)$. The vector $\nabla f(a)$ points into the direction in which $f$ increases most rapidly from

**Interpretation:** The gradient is a "list" of the changes in the value of the function if each variable separately changes infinitesimally: The first element tells us how much $f$ changes if $x_1$ changes a tiny bit, the second element tells us how much $f$ changes if $x_2$ changes a tiny bit etc. It gives the slopes of the tangent lines to the graph of $f$ at point $x$ in each of the coordinate directions.

The vector $\nabla f(a)$ also happens to point into the direction in which $f$ increases most rapidly from point $a$, i.e. $\nabla f(a)$ gives the direction of steepest ascent.

**Example:**
Let $f(x, y) = x^2 y^3$, and $a = (1, 2)$. Then

$$\nabla f = \begin{pmatrix} 2xy^3 \\ 3x^2 y^2 \end{pmatrix}$$

and

$$\nabla f(a) = \begin{pmatrix} 16 \\ 12 \end{pmatrix}$$

## 2.3 The Total Differential

The total differential of a function $f(x_1, x_2, \ldots, x_n)$ at a point $a = (x_1^*, x_2^*, \ldots, x_n^*)$ is

$$df = \frac{\partial f}{\partial x_1}(a) dx_1 + \frac{\partial f}{\partial x_2}(a) dx_2 + \ldots + \frac{\partial f}{\partial x_n}(a) dx_n.$$

**Interpretation:** The interpretation of the total differential is easy to see by looking at a tangent plane to the function $f(x, y)$. Let $dx$ and $dy$ be thought of as small deviations from the points $x^*$ and $y^*$ respectively. The total differential states that the change in the value of the function $f$ from this small deviation from $(x^*, y^*)$ to $(x^* + dx, y^* + dy)$ is the slope of the plane at $(x^*, y^*)$ in the $x$ direction times the deviation $dx$, plus the slope of the plane at $(x^*, y^*)$ in the $y$ direction times the deviation $dy$.

**Example:**
Given a utility function of the form $U(C, N) = \ln(C) + \ln(N)$, find the approximate change in utility by consuming an additional 0.1 units of $C$ and less 0.1 units of $N$. Assume the individual originally was consuming 0.5 units of each.

$$dU \approx \frac{1}{C}dC + \frac{1}{N}dN = \frac{1}{0.5}0.1 + \frac{1}{0.5}(-0.1) = 0$$

Note that the total differential is only valid with equality as $(dx, dy) \to (0,0)$. Otherwise it is only an approximation.

## 2.4 Second order partial derivatives

The second order partial derivative is just a partial derivative of a partial derivative. The derivative with respect to $x_i$ of the derivative with respect to $x_j$ of the function $f$, or $\frac{\partial}{\partial x_i}(\frac{\partial f}{\partial x_j})$, is denoted by $\frac{\partial^2 f}{\partial x_i \partial x_j}$.

**Example:**
Let $f(x,y) = x^2y^3$. Then

$$\frac{\partial^2 f}{\partial x^2} = 2y^3$$
$$\frac{\partial^2 f}{\partial y^2} = 6x^2y$$
$$\frac{\partial^2 f}{\partial x \partial y} = 6xy^2$$
$$\frac{\partial^2 f}{\partial y \partial x} = 6xy^2$$

Notice that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$.

**The Hessian Matrix**

The Hessian is merely a matrix of the second order partial derivatives of a function. Consider a function $f(x_1, \ldots, f_n)$. Its Hessian is given by.

$$\begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n} & \frac{\partial^2 f}{\partial x_2 \partial x_n} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix}.$$

Note that a Hessian matrix of $f$ is always a square symmetric matrix if $f \in C^2$ by Young's theorem (p.330 in SB), Clairaut's theorem, or Schwarz's theorem.

**Example**:
Let $f(x,y) = x^2y^3$. Then the Hessian is

$$\begin{pmatrix} 2y^3 & 6xy^2 \\ 6xy^2 & 6x^2y \end{pmatrix}.$$

**Problem:**

1. Find the Hessian of $\frac{1}{x_1} + \frac{1}{x_2} + e^{x_3}$.

2. Check the second order partial derivatives of

$$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$$

at $(0,0)$.

## 2.5 Chain Rule

Let $\mathbf{x}(t) = (x_1(t), \ldots, x_n(t))$. Also, let $f$ be some function of $\mathbf{x}(t)$. Then

$$\frac{df}{dt}(\mathbf{x}(t)) = \frac{\partial f}{\partial x_1} \cdot \frac{dx_1}{dt} + \cdots + \frac{\partial f}{\partial x_n} \cdot \frac{dx_n}{dt}.$$

**Example:**
Let $K$ (capital stock) and $L$ (labor stock) be functions of time, such that $\frac{dK}{dt} = sY - \delta K$ and $\frac{dL}{dt} = nL$. If $Y = K^\alpha L^{1-\alpha}$, what is $\frac{dY}{dt}$?

$$\frac{dY}{dt} = \frac{\partial Y}{\partial K} \cdot \frac{dK}{dt} + \frac{\partial Y}{\partial L} \cdot \frac{dL}{dt}$$

$$\frac{dY}{dt} = \alpha \left(\frac{L}{K}\right)^{1-\alpha} \cdot (sY - \delta K) + (1-\alpha)\left(\frac{K}{L}\right)^{\alpha} \cdot nL$$

$$\frac{dY}{dt} = \left\{ \alpha \left[ s\left(\frac{L}{K}\right)^{1-\alpha} - \delta \right] + (1-\alpha)\, n \right\} \cdot Y$$

**Problem:** (The Solow Growth Model)
Let $k = \frac{K}{AL}$, $n = \frac{dL/dt}{L}$, $g = \frac{dA/dt}{A}$, $\frac{dK}{dt} = sY - \delta K$, and $f(k) = \frac{Y}{AL}$. Find $\frac{dk}{dt}$.

## 2.6 Directional Derivatives

Assume a function $f(\mathbf{x})$, where $\mathbf{x} = (x_1, \ldots, x_n)$ is a point in $\mathbb{R}^n$. Now let there be a vector $\mathbf{v} = (v_1, \ldots, v_n)$, and a parameter $t$. The point of a directional derivative is to see how the value of the function $f(\mathbf{x})$ changes as we move along the vector $\mathbf{v}$ from the original point $\mathbf{x}$. A line can be drawn from $\mathbf{x}$ along $\mathbf{v}$ in $\mathbb{R}^n$ by constructing a point $\mathbf{x} + t \cdot \mathbf{v}$, and allowing $t$ to vary. Define a new function $g(t) = f(\mathbf{x} + t \cdot \mathbf{v}) = f(x_1 + t \cdot v_1, \ldots, x_n + t \cdot v_n)$, and derivate it with respect to $t$,

$$\frac{dg}{dt} = \frac{\partial f}{\partial x_1}(x_1 + t \cdot v_1, \ldots, x_n + t \cdot v_n) \cdot v_1 + \cdots + \frac{\partial f}{\partial x_n}(x_1 + t \cdot v_1, \ldots, x_n + t \cdot v_n) \cdot v_n.$$

Since we are interested in the change of $f$ at the original point $\mathbf{x}$, we let $t = 0$ to get

$$\frac{dg}{dt}(0) = \frac{\partial f}{\partial x_1}(x_1, \ldots, x_n) \cdot v_1 + \cdots + \frac{\partial f}{\partial x_n}(x_1, \ldots, x_n) \cdot v_n = [\nabla f(\mathbf{x})]' \cdot \mathbf{v}.$$

This is the directional derivative. It is merely the dot product of the total derivative at $\mathbf{x}$ with the vector $\mathbf{v}$. Other notiations are $Df_{\mathbf{x}} \cdot \mathbf{v}$ or $\frac{\partial f}{\partial \mathbf{v}}(\mathbf{x})$.

# 3 Derivatives of Multiple Functions of Several Variables

Instead of assuming we have one function $f$, say we now have $m$ equations which are each a function of $n$ variables:

$$y_1 = f_1(x_1, \ldots, x_n)$$

$$\vdots$$

$$y_m = f_m(x_1, \dots, x_n)$$

As noted above, we can denote this as a single function $f$ which is a function from $\mathbb{R}^n$ to $\mathbb{R}^m$. (Why? Because we can think of the system of equations as a box into which we throw $n$ inputs (the $x$'s) and get $m$ outputs (the $y$'s)).

Recall that the total derivative of any function $f$ is

$$Df = \begin{pmatrix} \frac{\partial f}{\partial x_1} & \dots & \frac{\partial f}{\partial x_n} \end{pmatrix}$$

How do we interpret $\frac{\partial f}{\partial x_1}$? Since $f$ is actually $m$ functions, we say that

$$\frac{\partial f}{\partial x_1} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} \\ \vdots \\ \frac{\partial f_m}{\partial x_1} \end{pmatrix}$$

Therefore, $Df$ is now an $n \times m$ matrix called the Jacobian matrix

$$Df(\mathbf{x}^*) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}^*) & \dots & \frac{\partial f_1}{\partial x_n}(\mathbf{x}^*) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{x}^*) & \dots & \frac{\partial f_m}{\partial x_n}(\mathbf{x}^*) \end{pmatrix}$$

where $\mathbf{x}^*$ is a particular value of $\mathbf{x} = (x_1, \dots, x_n)$.

**Example:**

Find the Jacobian of the following system:

$$W(X,Y) = \frac{X}{Y}$$

$$V(X,Y) = XY$$

Notice that

$$DW = (\frac{1}{Y}, -\frac{X}{Y^2})$$

and

$$DV = (Y, X),$$

so the Jacobian is

$$\begin{pmatrix} \frac{1}{Y} & -\frac{X}{Y^2} \\ Y & X \end{pmatrix}.$$

**Problem:** Find the Jacobian of the following function $F(Y_1, Y_2) : \mathbb{R}^2 \to \mathbb{R}^2$

$$X_1 = \sqrt{Y_1 Y_2},$$

$$X_2 = \sqrt{\frac{Y_2}{Y_1}}.$$

# 4 Integration over Several Variables

Let $f$ be a function of several variables. For simplicity we assume $f$ to be a function of two variables, $f : \mathbb{R}^2 \to \mathbb{R}$. The integral

$$\int_a^b f(x,y)dx$$

is simply computed by treating $f(x,y)$ as a function of $x$ only, that is, $y$ is treated as a constant. Note that the result of the integration will be a function of $y$.

**Example**:
Evaluate $\int_2^4 (x+y)dx$.

$$\int_2^4 (x+y)dx = (\frac{1}{2}x^2 + xy)|_2^4 = 6 + 2y$$

## 4.1 Double and Higher Order Integrals

In order to evaluate the double integral

$$\int_a^b \int_c^d f(x,y)dxdy$$

it is helpful to rewrite the integral as

$$\int_a^b \left( \int_c^d f(x,y)dx \right) dy$$

and evaluate the inside integral first. Then the outside integral can be evaluated to obtain the solution.

**Example:**
Evaluate $\int_1^2 \int_2^4 (x+y)dxdy$.

$$\int_1^2 \int_2^4 (x+y)dxdy = \int_1^2 \left( \int_2^4 (x+y)dx \right) dy = \int_1^2 \left( (\frac{1}{2}x^2 + xy)|_2^4 \right) dy = \int_1^2 (6+2y)\, dy = 6y + y^2|_1^2 = 9.$$

Triple integrals (and higher order integrals) are evaluated in the same manner as doubles; we first integrate with respect to one variable and work our way back to lower order integrals.

$$\int_a^b \int_c^d \int_e^f f(x,y,z)dxdydz = \int_a^b \left( \int_c^d \left( \int_e^f f(x,y)dx \right) dy \right) dz$$

## 4.2 Order of Integration

The limits of integration will only be scalar if we are integrating over a rectangle (or its equivalent in higher dimensions). In many cases, the simplicity of the order of integration does not matter.

**Example:**
Recall the double integral $\int_1^2 \int_2^4 (x+y)dxdy$ above which we evaluated with respect to $x$ first. Now evaluate with respect to $y$ first.

$$\int_2^4 \int_1^2 (x+y)dydx = \int_2^4 \left( \int_1^2 (x+y)dy \right) dx = \int_2^4 \left( (\frac{1}{2}y^2 + xy)|_1^2 \right) dy =$$

$$= \int_2^4 \left(\frac{3}{2} + x\right) dy = \frac{3}{2}y + \frac{1}{2}y^2|_2^4 = 3 + 6 = 9.$$

However, we are integrating with respect to variables that are also in the limits of integration, then different orders of integration influence the difficulty of the problem. To see this, consider the integral of $f(x,y)$ over the region $R = \{1 \leq y \leq 2,\ y \leq x \leq y^2\}$. By drawing the area of integration, we can easily see that integrating with respect to $x$ first is much easier. Notice that the region $R$ is equivalent to the region $S = \{1 \leq x \leq 2,\ x \leq y \leq \sqrt{x}\} \cup \{2 \leq x \leq 4,\ 2 \leq y \leq \sqrt{x}\}$. If I integrate with respect to $y$ first, I must split the integral into two parts, since the bounds of integration in the $x$ direction change at $x = 2$.

**Example:**
Evaluate $\int_1^2 \int_x^{2x} xy^2 dydx$ with respect to $y$ first:

$$\int_1^2 \int_x^{2x} xy^2 dydx = \int_1^2 (\frac{1}{3}xy^3|_x^{2x})dx = \int_1^2 \frac{7}{3}x^4 dx = \frac{7}{15}x^5|_1^2 = \frac{7}{15}(2^5 - 1^5) = \frac{217}{15}$$

Now evaluate with respect to $x$ first:

$$\int_1^2 \int_x^{2x} xy^2 dydx = \int_1^2 \int_1^y xy^2 dxdy + \int_2^4 \int_{\frac{y}{2}}^2 xy^2 dxdy = \int_1^2 \frac{1}{2}x^2y^2|_1^y dy + \int_2^4 \frac{1}{2}x^2y^2|_{\frac{y}{2}}^2 dy =$$

$$= \int_1^2 (\frac{1}{2}y^4 - \frac{1}{2}y^2)dy + \int_2^4 (2y^2 - \frac{y^4}{8})dy = (\frac{1}{10}y^5 - \frac{1}{6}y^3)|_1^2 + (\frac{2}{3}y^3 - \frac{1}{40}y^5)|_2^4 =$$

$$= \frac{31}{10} - \frac{7}{6} + \frac{112}{3} - \frac{992}{40} = \frac{217}{15}.$$

## 4.3 Transformations

Consider the integral $\int_a^b \int_c^d f(x,y)$. As we have seen previously, integrals are sometimes easier to evaluate when we substitute a variable $u$ in for a function of $x$ and $y$. Consider the following integral:

$$\int_2^3 \int_0^{x-2} \frac{1}{(x+y)(x-y)}$$

We can guess that making a substitution $u = x + y$ and $v = x - y$ would make the integration a lot simpler. However, there is more than meets the eye in making this transformation. Not only will the limits of integration change, but we must also change the function itself beyond just making the substitution.

In general, say we are given $f(x,y)$, a region of integration, and two transformation functions $u(x,y)$ and $v(x,y)$. In order to integrate using the transformation, we first solve the system of transformation functions for $x(u,v)$ and $y(u,v)$. Then me must compute the Jacobian matrix

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

and find the absolute value of its determinant. We will cover determinants later, but the absolute value of the determinant of this specific matrix is $\left|\frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \cdot \frac{\partial y}{\partial u}\right| = |J|$. We then go back to the original equation $f(x,y)$, plug in $x(u,v)$ and $y(u,v)$ to get a new equation $g(u,v)$ and multiply it by the Jacobian to get our new integrand

$$\int\int g(u,v)\,|J|\,dudv.$$

Finally, we must change the limits of integration. This is done simply by looking at the possible values $x$ and $y$ took on, and then seeing what that would correspond to in $u$ and $v$. For example, if $x, y \in (0,1)$ and $u = xy$ and $v = \frac{x}{y}$, we can see that $u \in (0,1)$ and $v \in (0, \infty)$. We then integrate over that region.

Note that this is the generalization of the "integration by substitution" technique discussed in the notes on single variable calculus. In the single variable case, the determinant of the Jacobian is simply the derivative of the transformation function. There is only one subtle difference: In the multi-dimensional case we use the absolute value of the determinant, in the single variable case we use the determinant including its sign.

**Example:** Transform $\int_2^3 \int_0^{x-2} \frac{1}{(x+y)(x-y)} dydx$ using $u = x + y$ and $v = x - y$.

Solving the system for $x$ and $y$ we get $x = \frac{u+v}{2}$, $y = \frac{u-v}{2}$. The Jacobian of this system is

$$\left| \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \right| = \frac{1}{2}$$

Plugging this into the original equation, we have

$$\int \int \frac{1}{(\frac{u+v}{2} + \frac{u-v}{2})(\frac{u+v}{2} - \frac{u-v}{2})} \cdot \frac{1}{2} dudv = \int \int \frac{1}{2uv} dudv$$

To find the new limits of integration, we notice that there are three boundaries to the original integral: $x \leq 3$, $y \geq 0$, $y \leq x - 2$. Substituting in for and $x$ and $y$ in each of these three, we get:

$$\frac{u+v}{2} \leq 3 \Rightarrow u + v \leq 6$$

$$\frac{u-v}{2} \geq 0 \Rightarrow u \geq v$$

$$\frac{u-v}{2} \leq \frac{u+v}{2} - 2 \Rightarrow v \geq 2$$

Plotting these three, we see that the area of integration in $u$, $v$ space is now an isosceles triangle with corners at $(2,2)$, $(3,3)$, and $(4,2)$. We can then set up the orders of integration so the integral now reads

$$\int_2^3 \int_v^{6-v} \frac{1}{2uv} dudv.$$

This is hard to integrate, so we will leave it for the interested student to do at home.

## 4.4 Leibnitz's Integral Rule

In order to differentiate under a definite integral, we use Leibnitz Rule:

$$\frac{d}{dy} \int_{a(y)}^{b(y)} f(x,y)dx = \int_{a(y)}^{b(y)} \frac{\partial f}{\partial y} dx + f[b(y), y] \frac{db}{dy} - f[a(y), y] \frac{da}{dy}$$

**Example:**

Find the derivative of $\int_0^1 x^2 y^2 dx$ with respect to $y$.

$$\frac{d}{dy} \int_0^1 x^2 y^2 dx = \int_0^1 2x^2 y dx.$$

**Example:**

Find the derivative of $\int_{2y}^{3y} x^2y^2dx$ with respect to $y$.

$$\frac{d}{dy}\int_{2y}^{3y} x^2y^2dx = \int_{2y}^{3y} 2x^2ydx + (3y)^2y^2 \cdot 3 - (2y)^2y^2 \cdot 2.$$

**Problem:** Differentiate $\int_{2y^{-2}}^{3y^{-2}} x^2y^2dx$ with respect to $y$.

# 5 Homework

In 1st semester micro, you will solve general equilibrium models. Sometimes when solving these models it is useful to see if utility functions are concave. One way of testing for concavity involves calculating the function's Hessian. Find the Hessian matrices of the following utility functions (these functions were used in previous homeworks and tests):

1. (Core Exam) $U(x_1, x_2) = \frac{2}{3}\sqrt{x_1} + \frac{1}{3}\sqrt{x_2}$
2. (Final Exam) $U(x_1, x_2) = x_1 + \frac{\delta}{\alpha} x_2^{\alpha}$
3. (Homework problem) $U(x_1, x_2, x_3) = -\frac{1}{x_1} + x_2 - \delta\frac{1}{x_3}$
4. (Midterm Exam) $U(x_1, x_2) = \frac{1}{3}\ln(x_1) + \frac{2}{3}\ln(x_2)$

The Slutsky Equation breaks changes in demand into income effects and substitution effects. Last semester for one of the homework problems, we were asked to calculate the Slutsky equation to find the substitution effect and the income effect. In the problem, the utility function was the same as in question 3 above, and it can be shown that the direct demand functions are

$$x_1(p_1, p_2, w) = \frac{w}{3p_1}$$

$$x_2(p_1, p_2, w) = \frac{2w}{3p_2}$$

Let the function $\mathbf{x}(p, w) : \mathbb{R}^3 \to \mathbb{R}^2$ be the combined demand function, where $p = (p_1, p_2)$. The Slutsky equation is as follows:

$$D_p\mathbf{x} + D_w\mathbf{x} \cdot \mathbf{x}' = D_p\mathbf{v},$$

where the subscripts denote which derivative it is respect to (Note: these are total derivatives, not directional derivatives) and $v$ is the indirect utility function. Find $D_p\mathbf{x}$ (the total effect), $D_w\mathbf{x} \cdot \mathbf{x}'$(the wealth effect), and use the above Slutsky equation to compute $D_p\mathbf{v}$ (the substitution effect). Hint: $D_p\mathbf{x}$ is a $2 \times 2$ matrix, $D_w\mathbf{x}$ is $1 \times 2$, and $\mathbf{x}$ is $1 \times 2$. This involves matrix multiplication, something we haven't covered yet, but that you should already know.

If the graph of a function $F : \mathbb{R}^3 \to \mathbb{R}^1$ lives in $\mathbb{R}^4$, in which space does the gradient $\nabla F$ live?

Evaluate the following integrals:

1. $\int_0^2 \int_0^{4-x^2} xydydx$
2. $\int_0^1 \int_{1-x}^{\sqrt{1-x^2}} x^2 ydydx$
3. $\int\int xdydx$ for the region bounded by $y = x$ and $y = 3 - x^2$
4. $\int_0^1 \int_y^1 x^2 \sin(xy)dxdy$

Sketch the following regions:

1. $1 \le x \le 2$, $2 \le y \le 7$
2. $0 \le x \le 2$, $\frac{x^2}{2} \le y \le x$

3. $1 \leq x \leq 2$, $x^2 \leq y \leq x+2$
4. $0 \leq x \leq 1$, $x^2 \leq y \leq x$

Integrate the following:

1. $f(x,y) = \sin(x^2)$ over the area $0 \leq x \leq 2$, $0 \leq y \leq \frac{x}{2}$ with respect to $x$ first
2. $f(x,y) = \sin(x^2)$ over the area $0 \leq x \leq 2$, $0 \leq y \leq \frac{x}{2}$ with respect to $y$ first

Differentiate the following:

1. $\int_x^1 \lambda e^{-\lambda xy} dy$ with respect to $x$.
2. $\int_x^{e^x} 2xe^{xy} dy$ with respect to $x$.

In 1st semester econometrics, you will be asked to integrate probability density functions (or p.d.f.s) in order to find the probability that certain events will occur. Sometimes it is necessary to transform the random variables. For example, if we are given the density functions for $X_1$ and $X_2$, and $Y_1$ and $Y_2$ as functions of $X_1$ and $X_2$, we can then transform this system to solve for the p.d.f.'s of $Y_1$ and $Y_2$. The following problems are taken from p.d.f.'s in the first semester econometrics textbook, section 3.7. For each of the following,

- Find $X_1$ and $X_2$ as functions of $Y_1$ and $Y_2$
- Find the determinant of the Jacobian
- Sketch the region of integration in terms of $X_1$ and $X_2$
- Sketch the region of integration in terms of $Y_1$ and $Y_2$
- Evaluate the new integral in terms of $Y_1$ and $Y_2$

1. $f(X_1, X_2) = e^{-X_1 - X_2}$ over the region $X_1 > 0$, $X_2 > 0$, where $Y_1 = \frac{X_1}{X_1 + X_2}$ and $Y_2 = X_1 + X_2$.
2. $f(X_1, X_2) = 8X_1X_2$ over the region $0 \leq X_1 \leq X_2 \leq 1$, where $Y_1 = \frac{X_1}{X_2}$ and $Y_2 = X_2$.

Hint: As a check, realize that all p.d.f.'s integrate to one. So the original integrals with respect to $X_1$ and $X_2$, as well as the transformed integral with respect to $Y_1$ and $Y_2$, should integrate to one. Try it!