Real Analysis Solutions[1]

Math Camp 2012

State whether the following sets are open, closed, neither, or both:

1. $\{(x, y) : -1 < x < 1, y = 0\}$ Neither
2. $\{(x, y) : x, y \text{ are integers}\}$ Closed
3. $\{(x, y) : x + y = 1\}$ closed
4. $\{(x, y) : x + y < 1\}$ open
5. $\{(x, y) : x = 0 \text{ or } y = 0\}$ closed

Prove the following:

1. Open balls are open sets

   Take any $y \in B(x, r)$. Define $r_2 = \frac{r-d(y,x)}{2}$. Let z be any point in $B(y, r_2)$. Then

   $$d(z, x) \leq d(z, y)+d(y, x) \leq r_2+d(y, x) = \left(\frac{r}{2} - \frac{1}{2}d(y, x) + d(y, x)\right) = \frac{r}{2}+\frac{1}{2}d(y, x) \leq \frac{1}{2}r+\frac{1}{2}r = r$$

   therefore $z \in B(x, r)$, then $B(y, r_2) \subset B(x, r)$. QED

2. Any union of open sets is open

   Let $U = U_1 \cup U_2 \cup ....$ (the union of sets U_i where there can be infinitely many), where U_i is open for all i. Take any $x \in U$, then $x \in U_i$ for some set U_i. Since U_i is open then $\exists r$ s.t. $B(x, r) \subset U_i$, but since by definition $U_i \subset U$, then we have that $B(x, r) \subset U$, and therefore U is open.

3. The finite intersection of open sets is open

   Let $U = U_1 \cap U_2 \cap U_3 \cap .... \cap U_k$ where $U_1, U_2, ..., U_k$ are open sets. Take any $x \in U$, then $x \in U_i$ for all $i \in \{1, 2, ..., k\}$. Since U_i is open, there exists r_i such that $B(x, r_i) \subset U_i$. Let $r \equiv \min\{r_1, ..., r_k\}$, then $B(x, r) \subset B(x, r_i) \subset U_i$ for all $i \in \{1, 2..., k\}$, therefore $B(x, r) \subset U$, therefore U is open. QED.

4. Any intersection of closed sets is closed

5. The finite union of closed sets is closed

   For 4 and 5 use the fact that $U = U_1 \cap U_2 \cap .... \Leftrightarrow U^c = U_1^c \cup U_2^c \cup .....$ and that $U = U_1 \cup U_2 \cup .... \cup U_k \Leftrightarrow U^c = U_1^c \cap U_2^c \cap .... \cap U_k^c$ and then use the proofs for part 2 and 3.

6. Let f and g be functions from $\mathbb{R}^k$ to $\mathbb{R}^m$ which are continuous at x. Then $h = f - g$ is continuous at x.

[1] If you find any typo please email me: Maria_Jose_Boccardi@Brown.edu

Use the alternative definition for continuity for sequences. Then we have that: take any sequence $\{x_i\}_{i\in\mathbb{N}} \subset \mathbb{R}^k$ such that $\{x_i\}_{i=1}^{\infty} \to x$. Then we need to show that $h(x_i) \to h(x)$ as $i \to \infty$. By the definition of h we have that $h(x_i) = f(x_i) - g(x_i)$, therefore

$$\lim_{i\to\infty} h(x_i) = \lim_{i\to\infty} f(x_i) - g(x_i) = \lim_{i\to\infty} f(x_i) - \lim_{i\to\infty} g(x_i) = f(x) - g(x)$$

when in the second to last step we use the property of limits and in the last step the fact that f and g are continuous.

7. Let f and g be functions from $\mathbb{R}^k$ to $\mathbb{R}^m$ which are continuous at x. Then $h = fg$ is continuous at x.

   Analogous to previous case using the property that if $z_n = x_n y_n$ then $\lim_{n\to\infty} z_n = [\lim_{n\to\infty} x_n][\lim_{n\to\infty} y_n]$

Find the greatest lower bound and the least upper bound of the following sequences. Also, prove whether they are convergent or divergent:

1. $\{x_n\}_{i=1}^{\infty} = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots\right\}$

   Greatest lower bound is $\frac{1}{2}$ and the least upper bound is 1

   Claim 0.1 $\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{n}{n+1} = 1$

   Proof. Fix $\varepsilon > 0$. Let $K > \frac{1}{\varepsilon}$ take any $n \geq K$ then

$$\left|1 - \frac{n}{n+1}\right| = \frac{1}{n+1} < \frac{1}{n} < \frac{1}{K} < \varepsilon$$

   ■

2. $\{x_n\}_{i=1}^{\infty} = \{-1, 1, -1, 1, \dots\}$

   Greatest lower bound is -1 and the least upper bound is 1

   Claim 0.2 $\lim_{n\to\infty} x_n = \lim_{n\to\infty}(-1)^n \nexists$. *That is* $\{x_n\}$ *diverges.*

   Proof. Suppose, by contradiction, that $\{x_n\}$ has a limit point L. Take $\varepsilon = \frac{1}{4}$ then there exists K such that $d(x_n, L) < \frac{1}{4}$ for all $n > K$. Therefore $d(x_n, x_{n+1}) \leq d(x_n, L) + d(x_{n+1}, L) \leq \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ for all $n > K$. But $d(x_n, x_{n+1}) = |1 - (-1)| = 2$ for all $n \in \mathbb{N}$, therefore we have a contradiction, then $\{x_n\}$ does not have a limit point. ■

3. $\{x_n\}_{i=1}^{\infty} = \left\{-\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, -\frac{5}{6}, \dots\right\}$

   Greatest lower bound is -1 and the least upper bound is 1

   Claim 0.3 $\lim_{n\to\infty} x_n = \lim_{n\to\infty}(-1)^n \frac{n}{n+1} \nexists$, *that is* $\{x_n\}$ *diverges*

   Proof. Analogous to previous one. ■

Prove the following:

1. A sequence can only have at most one limit.

   Suppose, by contradiction, that $\{x_n\}$ has two limits $L_1 \neq L_2$. Choose $\varepsilon = \frac{d(L_1,L_2)}{4}$. Then there exist $K_1, K_2 \in \mathbb{N}$ such that $d(x_n, L_1) < \varepsilon$ for all $n \geq K_1$ and $d(x_n, L_2) < \varepsilon$ for all $n \geq K_2$. Define $K \equiv \max\{K_1, K_2\}$. Then

   $$d(L_1, L_2) \leq d(L_1, x_n) + d(x_n, L_2) \leq \varepsilon + \varepsilon = 2\varepsilon$$

   but given the way that ε was defined we have that $d(L_1, L_2) = 4\varepsilon > 2\varepsilon$, therefore we have a contradiction and it must be the case that $L_1 = L_2$

2. If $\{x_n\}_{n=1}^{\infty} \to x$ and $\{y_n\}_{n=1}^{\infty} \to y$, then $\{x_n + y_n\}_{n=1}^{\infty} = x + y$.

   Fix $\varepsilon > 0$, then $\exists K_1, K_2 \in \mathbb{N}$ st $|x_n - x| < \frac{\varepsilon}{3}$ for all $n \geq K_1$ and $|y_n - y| < \frac{\varepsilon}{3}$ for all $n \geq K_2$. Define $K \equiv \max\{K_1, K_2\}$ then we have that

   $$|(x_n + y_n) - (x + y)| = |(x_n - x) + (y_n - y)| \leq |x_n - x| + |y_n - y| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3} < \varepsilon$$

   for all $n \geq K$ therefore we have that $\{x_n + y_n\} \to x + y$

3. A sequence of vectors in $\mathbb{R}^N$ converges iff all the component sequences converge in $\mathbb{R}$.

   We are going to show this only for the Euclidean distance, that is $d(x, y) = \sqrt{\sum_{i=1}^{N}(x_i - y_i)^2}$. We need to prove both statments "if" and ""only if".
   Proof. [$\Rightarrow$] Suppose that $\{x_n\} \to x \in \mathbb{R}^N$, then we need to show that $\{(x_i)_n\} \to x_i \in \mathbb{R}$ for all $i \in \{1, 2, .., N\}$

   Fix $\varepsilon > 0$ then $\exists K$ s.t $d(x_n, x) < \varepsilon$ for all $n \geq K$, where $d(x_n, x) = \sqrt{\sum_{i=1}^{N}((x_i)_n - x_i)^2}$, therefore it has to be the case that $((x_i)_n - x_i)^2 < \varepsilon^2$ for all $i \in \{1, .., N\}$ for all $n \geq K$, which in turn implies that $|(x_i)_n - x_i| < \varepsilon$ for all $i \in \{1, .., N\}$ for all $n \geq K$ that is $\{(x_i)_n\} \to x_i$ for all $i \in \{1, .., N\}$. ■
   Proof. [$\Leftarrow$] Suppose that $\{(x_i)_n\} \to x_i \in \mathbb{R}$ for all $i \in \{1, 2, .., N\}$, We want to show that $\{x_n\} \to x \in \mathbb{R}^N$.

   Fix $\varepsilon > 0$ for all $i \in \{1, ..., N\}$ there exists K_i such that $|(x_i)_n - x_i| < \frac{\varepsilon}{\sqrt{N}}$ for all $n \geq K_i$. Define $K \equiv \max\{K_1, ..., K_N\}$, then

   $$d(x_n, x) = \sqrt{\sum_{i=1}^{N}((x_i)_n - x_i)^2} \leq \sqrt{N\frac{\varepsilon^2}{N}} = \varepsilon$$

   for all $n \geq K$, and therefore $\{x_n\} \to x$. QED ■

4. The sequence $\{x_n\}_{n=1}^{\infty} = \left\{(1, \frac{1}{2}), (1, \frac{1}{3}), (1, \frac{1}{4}), \dots\right\}$ converges to $(1, 0)$.

   It is straightforward using the result from 3 and the fact that $\{x_n\} = \{\frac{1}{n}\} \to 0$.

5. The sequence $\{x_n\}_{n=1}^{\infty} = \left\{(\frac{1}{2}, \frac{1}{2}), (\frac{2}{3}, \frac{1}{3}), (\frac{3}{4}, \frac{1}{4}), \dots\right\}$ converges to $(1, 0)$.

   Idem previous exercises using also the result from part 1 of the previous exercise.