Unconstrained Optimization Solutions[1]

Math Camp 2012

1. For each function, determine whether it definitely has a maximum, definitively does not have a maximum, or that there is not enough information to tell, using the Weierstrass Theorem. If it definitely has a maximum, prove that this is the case.

   (a) $f : \mathbb{R} \to \mathbb{R},\ f(x) = x$. We cannot used the Weierstrass theorem given that the domain is not compact (not bounded). By looking at the function we know that there is not a local nor global maximum for this function in the real numbers.

   (b) $f : [-1, 1] \to \mathbb{R},\ f(x) = x$ By Weierstrass theorem we know that there exists a global maximum (in $[-1, 1]$), since the domain is compact and the function is continuous. In particular the maximum is attained at $x = 1$.

   (c) $f : (-1, 1) \to \mathbb{R},\ f(x) = x$ We cannot used the Weierstrass theorem given that the domain is not compact (not closed). By looking at the function we know that there is not a local nor global maximum in the domain, since f is strictly increasing therefore the supremum 1 is not attained in the set.

   (d) $f : [-1, 1] \to \mathbb{R},\ f(x) = \begin{cases} 0 \text{ if } x = 1 \\ x \text{ otherwise} \end{cases}$ We cannot use Weierstrass since the function is not continuous. In any case by looking at the function we know that there is not a maximum in the domain, since the discontinuity happens at $x = 1$.

   (e) $f : \mathbb{R}_{++} \to \mathbb{R},\ f(x) = \begin{cases} 1 \text{ if } x = 5 \\ 0 \text{ otherwise} \end{cases}$ We cannot use Weierstrass since the function is not continuous. From the definition of the function we can see that the global maximum (not strict) is attained at $x = 5$

2. Consider the standard utility maximization problem

$$\max_{x \in B(p,I)} U(x), \text{ where } B(p, I) = \{x \in \mathbb{R}^n_+ |\ p \cdot x \leq I\}$$

   Prove a solution exists for any $U(x)$ continuous, $I > 0$ and $p \in \mathbb{R}^n_{++}$. Show a solution may not necessarily exist if $p \in \mathbb{R}^n_+$.

   We just need to show that $B(p, I)$ is a compact set, if so, then we are in the conditions of the Weierstrass theorem and therefore we know that there exists a solution to the utility maximization problem.

   B(p,I) is bounded since $B(p, I) \subset B\left(0, \frac{I}{p_{min}}\right)$, where $p_{min} \equiv \min_i\{p_i\}$

   Claim 0.1 ***B(p,I) is closed*** *since $B(p, I)^C$ is open*

   Proof. The easy way to prove it is to show that $B(p, I)^C = \mathbb{R}^n_+ \cap B^*(p, I)$ where $B^*(p, I) = \{x \in \mathbb{R}^n |\ p \cdot x > I\}$ where by definition $\mathbb{R}^n_+$ is open in $\mathbb{R}^n_+$. Then we just need to prove that

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$B^*(p,I)$ is open. We need to prove that for all $x \in B^*(p,I)$ there exists some $r > 0$ such that $B(x,r) \subset B^*(p,I)$. Since we know that $x \in B^*(p,I)$, then we know that $px > I$. Therefore consider, if we are considering the Euclidean metric consider $r \equiv \min_i\{\frac{px-I}{2p_i}\}$, then we have that $B(x,r) \subset B^*(p,I)$. ■

Additionally we have that if at least one of the prices is zero the argument breaks down. For example let's imagine that $p_1 = 0$, then any amount of x_1 is affordable, and therefore the set $B(p,I)$ is not bounded.

3. Search for local maxima and minima in the following functions. More specifically, find the points where $DF(\mathbf{x}) = 0$, and then classify then as a local maximum, a local minimum, definitely not a maximum or minimum, or can't tell. Also, check whether the functions are concave, convex, or neither. The answers (except for the concavity/convexity part) are found in the back of Simon and Blume, Exercises 17.1 - 17.2.

(a) $F(x,y) = x^4 + x^2 - 6xy + 3y^2$

$$DF(x) = \begin{bmatrix} 4x^3 + 2x - 6y & -6x + 6y \end{bmatrix}$$

therefore we have that the critical points are given by $(0,0)$, $(1,1)$ and $(-1,-1)$.

$$D^2F(x) = \begin{bmatrix} 12x^2 + 2 & -6 \\ -6 & 6 \end{bmatrix}$$

So we have that

$$D^2F(0,0) = \begin{bmatrix} 2 & -6 \\ -6 & 6 \end{bmatrix}$$

then $(0,0)$ is a saddle point.

$$D^2F(1,1) = \begin{bmatrix} 14 & -6 \\ -6 & 6 \end{bmatrix}$$

since the first order leading principal minor is 14 and the second order leading principal minor is $14 \times 6 - 6 \times 6 = 6(14-6) > 0$, then the hessian is positive definite, therefore $(1,1)$ is a local minimum. Finally

$$D^2F(-1,-1) = \begin{bmatrix} 14 & -6 \\ -6 & 6 \end{bmatrix}$$

since the first order leading principal minor is 14 and the second order leading principal minor is $14 \times 6 - 6 \times 6 = 6(14-6) > 0$, then the hessian is positive definite, therefore $(-1,-1)$ is a local minimum.

(b) $F(x,y) = x^2 - 6xy + 2y^2 + 10x + 2y - 5$

Done in class.

(c) $F(x,y) = xy^2 + x^3y - xy$

Done in class.

(d) $F(x,y) = 3x^4 + 3x^2y - y^3$

$$DF(x) = \begin{bmatrix} 12x^3 + 6xy & 3x^2 - 3y^2 \end{bmatrix}$$

therefore we have that the critical points are given by $(0,0)$, $\left(-\frac{1}{2}, -\frac{1}{2}\right)$ and $\left(\frac{1}{2}, -\frac{1}{2}\right)$.

$$D^2F(x) = \begin{bmatrix} 36x^2 + 6y & 6x \\ 6x & -6y \end{bmatrix}$$

So we have that

$$D^2F(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

therefore we cannot tell.

$$D^2F\left(-\frac{1}{2},-\frac{1}{2}\right) = \begin{bmatrix} 6 & -3 \\ -3 & 3 \end{bmatrix}$$

since the first order leading principal minor is 6 and the second order leading principal minor is $6 \times 3 - 3 \times 3 = 9 > 0$, then the hessian is positive definite, therefore $(1,1)$ is a local minimum. Finally

$$D^2F\left(\frac{1}{2},-\frac{1}{2}\right) = \begin{bmatrix} 6 & 3 \\ 3 & 3 \end{bmatrix}$$

since the first order leading principal minor is 6 and the second order leading principal minor is $9 > 0$, then the hessian is positive definite, therefore $\left(\frac{1}{2},-\frac{1}{2}\right)$ is a local minimum.

(e) $F(x,y,z) = x^2 + 6xy + y^2 - 3yz + 4z^2 - 10x - 5y - 21z$

$$DF(x) = \begin{bmatrix} 2x+6y-10 & 6x+2y-3z-5 & -3y+8z-21 \end{bmatrix}$$

therefore we have that the critical point is given by $(2,1,3)$.

$$D^2F(x) = \begin{bmatrix} 2 & 6 & 0 \\ 6 & 2 & -3 \\ 0 & -3 & 8 \end{bmatrix}$$

so we have that the first order leading principal minor is 2, the second order principal minor is -32, therefore $(2,1,3)$ is a saddle point.

(f) $F(x,y,z) = \left(x^2 + 2y^2 + 3z^2\right) e^{-\left(x^2+y^2+z^2\right)}$

$$DF(x) = \begin{bmatrix} 2xe^{-\left(x^2+y^2+z^2\right)} + (-2x)\left(x^2+2y^2+3z^2\right)e^{-\left(x^2+y^2+z^2\right)} \\ 4ye^{-\left(x^2+y^2+z^2\right)} + (-2y)\left(x^2+2y^2+3z^2\right)e^{-\left(x^2+y^2+z^2\right)} \\ 6ze^{-\left(x^2+y^2+z^2\right)} + (-2z)\left(x^2+2y^2+3z^2\right)e^{-\left(x^2+y^2+z^2\right)} \end{bmatrix}'$$

$$DF(x) = \begin{bmatrix} 2xe^{-\left(x^2+y^2+z^2\right)}\left(1-\left(x^2+2y^2+3z^2\right)\right) \\ 2ye^{-\left(x^2+y^2+z^2\right)}\left(2-\left(x^2+2y^2+3z^2\right)\right) \\ 2ze^{-\left(x^2+y^2+z^2\right)}\left(3-\left(x^2+2y^2+3z^2\right)\right) \end{bmatrix}'$$

The first row of the Hessian is given by

$$D^2F(x)_1 = \begin{bmatrix} \left(2-6x^2-4y^2-6z^2\right)e^{-\left(x^2+y^2+z^2\right)} + (-2x)2x\left(1-\left(x^2+2y^2+3z^2\right)\right)e^{-\left(x^2+y^2+z^2\right)} \\ (-8yx)\,e^{-\left(x^2+y^2+z^2\right)} + (-2y)2x\left(1-\left(x^2+2y^2+3z^2\right)\right)e^{-\left(x^2+y^2+z^2\right)} \\ (-12zx)\,e^{-\left(x^2+y^2+z^2\right)} + (-2z)2x\left(1-\left(x^2+2y^2+3z^2\right)\right)e^{-\left(x^2+y^2+z^2\right)} \end{bmatrix}'$$

The second row of the Hessian is given by

$$D^2F(x)_2 = \begin{bmatrix} (-8yx)\,e^{-\left(x^2+y^2+z^2\right)} + (-2z)2x\left(1-\left(x^2+2y^2+3z^2\right)\right)e^{-\left(x^2+y^2+z^2\right)} \\ \left(4-2x^2-12y^2-6z^2\right)e^{-\left(x^2+y^2+z^2\right)} + (-2y)2y\left(2-\left(x^2+2y^2+3z^2\right)\right)e^{-\left(x^2+y^2+z^2\right)} \\ (-12zy)\,e^{-\left(x^2+y^2+z^2\right)} + (-2z)2y\left(2-\left(x^2+2y^2+3z^2\right)\right)e^{-\left(x^2+y^2+z^2\right)} \end{bmatrix}'$$

The third row of the Hessian is given by

$$D^2F(x)_3 = \begin{bmatrix} (-12zx)\,e^{-\left(x^2+y^2+z^2\right)} + (-2z)2x\left(1-\left(x^2+2y^2+3z^2\right)\right)e^{-\left(x^2+y^2+z^2\right)} \\ (-12zy)\,e^{-\left(x^2+y^2+z^2\right)} + (-2z)2y\left(2-\left(x^2+2y^2+3z^2\right)\right)e^{-\left(x^2+y^2+z^2\right)} \\ \left(6-2x^2-2y^2-6z^2\right)e^{-\left(x^2+y^2+z^2\right)} + (-2z)2z\left(2-\left(x^2+2y^2+3z^2\right)\right)e^{-\left(x^2+y^2+z^2\right)} \end{bmatrix}'$$

So the critical points are going to be defined by the following three conditions

(a) $x = 0$ or $1 - \left(x^2 + 2y^2 + 3z^2\right) = 0$

(b) $y = 0$ or $2 - \left(x^2 + 2y^2 + 3z^2\right) = 0$

(c) $z = 0$ or $3 - \left(x^2 + 2y^2 + 3z^2\right) = 0$