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Deterministic debt cycles in open economies with flow collateral constraints ☆

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Abstract

This paper establishes the existence of endogenous cycles in infinite-horizon open economy models with a flow collateral constraint. It shows that for plausible parameter configurations, the economy has a unique equilibrium exhibiting deterministic cycles in which periods of debt growth are followed by periods of debt deleveraging. In particular, three-period cycles exist, which implies by the Li-Yorke Theorem the presence of cycles of any periodicity and chaos. The paper also shows that deterministic cycles are absent in the Ramsey optimal allocation providing a justification for macroprudential policies even in the absence of uncertainty.

JEL classification: F41; F38; E32; H23

Keywords: Deterministic cycles; Collateral constraints; Capital controls

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1. Introduction

Financial frictions have been shown to amplify the business cycle. This paper argues that they can also be its engine. It studies a model of an open economy with a flow collateral constraint, whereby external debt is limited by the value of tradable and nontradable income. This environment has been extensively used to shed light on important issues in open economy macroeconomics such as inefficient credit booms, overborrowing, and sudden stops. However, the related literature has limited attention to economies driven by exogenous stochastic disturbances. The contribution of the present paper is to show that the mere presence of the financial friction can engender cyclical fluctuations. To highlight this result we abstract from any source of uncertainty and characterize perfect foresight equilibria. Furthermore, we focus on parameterization for which the equilibrium is unique.

The first result of the paper is a full analytical characterization of the debt policy function (i.e., the choice of current debt as a function of the level of past debt) in infinite-horizon environments in which agents are impatient ($\beta(1+r) < 1$) and lines of credit are tied to income. These two features are defining elements of the literature to which this paper contributes. To the best of our knowledge, this is the first paper to achieve this task. Although numerical characterizations in stochastic environments do exist. The paper shows that the debt policy function is continuous but nonmonotonic. Importantly, it shows that the maximum of the debt policy function lies above the 45-degree line and occurs at a level of debt below the steady state, that is, below the largest constant level of debt that satisfies the collateral constraint and guarantees positive consumption. This characteristic of the debt policy function gives rise to endogenous equilibrium dynamics in which the economy oscillates around the steady state level of debt and suffers from recurring inefficient credit booms followed by debt deleveraging.

The second contribution of the paper is to show that for conditions that obtain under plausible calibrations, the aforementioned oscillatory dynamics are periodic, which means that the economy perpetually fluctuates around the steady state without ever converging to it. The economy exhibits cycles of periodicity three. By the Li and Yorke (1975) theorem, this implies the existence of cycles of any periodicity and chaos. The endogenous cycles identified in this paper share a number of features of business cycles observed in emerging market economies. In particular, during the expansionary phase of the cycle, external credit grows, domestic absorption expands, the real exchange rate appreciates, and the current account deteriorates. At some point, the financial constraint binds, the credit boom comes to a stop, there is widespread debt deleveraging, the real exchange rate depreciates, and the current account experiences a reversal.

The emergence of endogenous cycles has to do with three key features of the class of models to which this paper belongs. One is that agents are impatient in the sense that their subjective discount rate exceeds the market discount rate ($\beta(1+r) < 1$). This feature drives agents to front load consumption. Absent financial frictions, household debt would rise monotonically and approach the natural debt limit. The second is a financial friction taking the form of a collateral constraint. The third feature of the model that is key for the emergence of endogenous cycles is the well known fact that when collateral depends on equilibrium prices, the collateral constraint creates a pecuniary externality. In the present model collateral is the sum of tradable and nontradable income so that the relative price of nontradables in terms of tradables affects the value of collateral measured in units of tradable goods. By this externality, agents fail to internalize the full costs of temporarily borrowing beyond the maximum level of debt that is sustainable in the long run. In particular, they fail to see that their individual borrowing, in the aggregate, fuels the credit boom by raising the value of collateral through real exchange rate appreciation and that their deleveraging by depreciating the real exchange rate exacerbates the credit crunch. Debt deleveraging has a cleansing effect, as debt levels must fall significantly below the level that is sustainable in the long run. At this point, impatient consumers feeling financially stronger embark on another credit boom and the story repeats itself.

The third result of the paper is to show that endogenous debt cycles are inefficient in the sense that they imply greater fluctuations in consumption than is socially optimal. We characterize analytically the debt policy function of the Ramsey planner. This characterization is also novel as only numerical approximations for stochastic economies are presented in the related literature. As is well known, the Ramsey planner behaves like an individual who becomes more patient in periods in which the collateral constraint is slack in the current period but binding in the next. Thus she puts more weight on the future costs of deleveraging than do private households. We present conditions under which it is optimal for the Ramsey planner to eliminate endogenous cycles altogether.

Finally, we characterize the associated optimal capital control policy and show that the planner puts capital control taxes into place when next-period debt in the laissez-faire economy exceeds the level of debt that is sustainable in the long run. This result is a refinement of an existing one in stochastic versions of the present economy, namely, that the Ramsey planner imposes capital controls in the current period when the probability that the collateral constraint will bind under the optimal allocation in the next period is strictly positive. This finding shows that capital controls can be optimal in deterministic environments.

This paper is related to a large and growing literature on financial constraints in open economy models. The type of flow collateral constraint we study was introduced in open economy models by Mendoza (2002) to understand sudden stops caused by fundamental shocks. The pecuniary externality that emerges in this framework and the consequent room for macroprudential policy is studied in Korinek (2011), Lorenzoni (2008), Bianchi (2011), Benigno et al. (2013, 2016), Schmitt-Grohé and Uribe (2017a, 2017b), Dávila and Korinek (2018), and Jeanne and Korinek (2019), among others. Multiple equilibria have been identified in models with stock collateral constraints, where debt is limited by the value of capital or land, by Jeanne and Korinek (2019) and Schmitt-Grohé and Uribe (2017a) and in models with flow collateral constraints, where debt is limited by income, in Schmitt-Grohé and Uribe (forthcoming). By contrast, the present paper focuses on parameterizations for which the equilibrium is unique and characterizes conditions under which there exist deterministic cycles taking the form of periodic equilibria or chaos.

The paper is also related to a closed-economy literature showing that financial frictions can give rise to endogenous instability in both infinite-horizon and overlapping-generations economies. Benhabib et al. (2016) show the existence of chaotic equilibrium dynamics when the financial friction takes the form of limited enforcement in the banking sector. Woodford (1989) shows that periodic equilibria and chaos can occur when the financial friction takes the form of market segmentation whereby workers are hand-to-mouth consumers and firms finance investment from retained earnings. Beaudry et al. (2020) characterize periodic equilibria in a New Keynesian model with consumer default risk and bank monitoring costs. Suarez and Sussman (1997) show the possibility of boom bust cycles in models with asymmetric information in financial markets and Dong and Xu (2020) in a model with banks and financially constrained heterogeneous firms. Matsuyama (2007) emphasizes agency problems in credit markets as a source of cycles. Azariadis et al. (2016) show the existence of self-fulfilling credit cycles in a model with unsecured corporate debt, Benhabib et al. (2018) in an economy with adverse selection in credit markets, and Cui and Kaas (forthcoming) in a model with corporate default. Gorton and Ordoñez (2014, 2020) and Chousakos et al. (2020) analyze overlapping generations frameworks in which financial constraints are informational in nature and information and productivity comove in such a way as to create credit-led boom bust cycles. Gu et al. (2013) show that endogenous credit cycles may arise in an economy with endogenous debt constraints due to limited commitment to repay. Martin and Ventura (2012) and Miao and Wang (2018) introduce financial frictions in models of rational bubbles to generate booms and busts in net worth and asset prices.

The remainder of the paper is organized as follows. Section 2 presents the model. Section 3 shows that in equilibrium the financial constraint must bind in an infinite number of periods and characterizes the steady state of the economy. Section 4 provides an analytical characterization of the debt policy function. Section 5 derives conditions under which deterministic cycles exist. Section 6 characterizes the Ramsey allocation and establishes conditions under which it is optimal to eliminate endogenous cycles. Section 7 concludes.

2. The model

This section presents a model of an open economy with tradable and nontradable goods in which debt is limited by a fraction of the value of tradable and nontradable income. This collateral constraint introduces a pecuniary externality because the price of nontradables, which affects the value of the nontradable component of income, is taken as exogenous by individual agents, but is endogenously determined in equilibrium. The present formulation is a workhorse model in the sudden stop literature. To isolate the role of financial frictions in generating endogenous business cycles, the model abstracts from any source of uncertainty.

Consider an open economy populated by a large number of identical households with preferences of the form

$$\sum_{t=0}^{\infty} \beta^t U(c_t), \tag{1}$$

where c_t denotes consumption in period t, $U(\cdot)$ denotes an increasing and concave period utility function, and $\beta \in (0, 1)$ denotes the subjective discount factor. The period utility function takes the CRRA form

$$U(c) = \frac{c^{1-\sigma} - 1}{1-\sigma},$$

with $\sigma > 0$. Consumption is assumed to be a composite of tradable and nontradable goods, taking the CES form

$$c_t = A(c_t^T, c_t^N) \equiv \left[a {c_t^T}^{1-1/\xi} + (1-a) {c_t^N}^{1-1/\xi} \right]^{1/(1-1/\xi)}, \tag{2}$$

with $\xi > 0$, $a \in (0, 1)$, and where c_t^T denotes consumption of tradables in period t and c_t^N denotes consumption of nontradables in period t. Households are assumed to have access to a one-period internationally-traded bond denominated in terms of tradable goods, which pays the interest rate $r > 0$. The household's sequential budget constraint is given by

$$c_t^T + p_t c_t^N + d_t = y^T + p_t y^N + \frac{d_{t+1}}{1+r}, \tag{3}$$

where p_t denotes the relative price of nontradables in terms of tradables (or the real exchange rate), d_t denotes the amount of debt assumed in period $t-1$ and due in period t, and $y^T, y^N > 0$ denote the endowments of tradables and nontradables, respectively. Households are subject to the standard no-Ponzi-game constraint $\lim_{t\to\infty}(1+r)^{-t} d_t \le 0$.

The collateral constraint takes the form

$$d_{t+1} \leq \kappa(y^T + p_t y^N), \tag{4}$$

where $\kappa > 0$ is a parameter. The pecuniary externality arises because of the presence of the relative price of nontradables, p_t, on the right-hand side of the collateral constraint (4). Each individual household takes p_t as exogenously determined, even though, collectively, the absorption of goods by households is a determinant of this relative price.

The following assumption makes the collateral constraint economically relevant by ruling out an equilibrium in which it never binds:

Assumption 1. The parameters κ and r satisfy

$$\kappa < \frac{1+r}{r}. \tag{5}$$

As will become clear shortly, this assumption says that the collateral constraint is violated at the natural debt limit—when debt is so high that the entire endowment of tradables must be devoted to pay interest and tradable consumption is nil.

A key assumption in the related literature, which we also maintain here, is that households discount future period utilities at a rate higher than the market discount rate:

Assumption 2. $\beta(1+r) < 1$.

Households choose sequences $c_t^T > 0$, $c_t^N > 0$, $c_t > 0$, and d_{t+1} to maximize (1) subject to (2)-(4), taking as given the path of the real exchange rate, p_t, and the initial debt position, d_0. The first-order conditions of this problem are (2)-(4),

$$U'(A(c_t^T, c_t^N))A_1(c_t^T, c_t^N) = \lambda_t, \tag{6}$$

$$\left(\frac{1}{1+r} - \mu_t\right)\lambda_t = \beta\lambda_{t+1}, \tag{7}$$

$$p_t = \frac{1-a}{a}\left(\frac{c_t^T}{c_t^N}\right)^{1/\xi}, \tag{8}$$

$$\mu_t \geq 0, \tag{9}$$

$$\mu_t\left[d_{t+1} - \kappa(y^T + p_t y^N)\right] = 0, \tag{10}$$

and the transversality condition

$$\lim_{t\to\infty} \frac{d_t}{(1+r)^t} = 0, \tag{11}$$

where $\beta^t\lambda_t$ and $\beta^t\lambda_t\mu_t$ denote the Lagrange multipliers on the sequential budget constraint (3) and the collateral constraint (4), respectively.

In equilibrium, the market for nontradables must clear, that is, $c_t^N = y^N$ for all t. Combining this market clearing condition with the household's sequential budget constraint, equation (3), we obtain the economy's resource constraint

$$c_t^T + d_t = y^T + \frac{d_{t+1}}{1+r}. \tag{12}$$

An equilibrium is then a set of sequences $\{c_t^T, \lambda_t, \mu_t, p_t, d_{t+1}\}_{t=0}^{\infty}$ satisfying the collateral constraint (4), the Euler equation (7), the nonnegativity constraint (9), the complementary slackness condition (10), the transversality condition (11), the resource constraint (12), and

$$U'(A(c_t^T, y^N))A_1(c_t^T, y^N) = \lambda_t, \tag{13}$$

$$p_t = \frac{1-a}{a}\left(\frac{c_t^T}{y^N}\right)^{1/\xi}, \tag{14}$$

and

$$c_t^T > 0, \tag{15}$$

given the initial level of debt, d_0.

3. The recurrent nature of a binding collateral constraint

A property of the present economy that is important for establishing the existence of endogenous cycles is that an equilibrium in which the collateral constraint never binds does not exist. To see this, suppose, contrary to the claim, that the collateral constraint never binds in equilibrium, that is, (4) always holds with a strict inequality. By (10) it then must be the case that $\mu_t = 0$ for all $t \geq 0$. In this case, the Euler equation (7) and the assumption that $\beta(1+r) < 1$ imply that λ_t converges to ∞. This in turn implies that c_t^T converges to 0, which follows from equation (13) and the assumed functional forms for $U(\cdot)$ and $A(\cdot,\cdot)$. The resource constraint (12) then implies that in the limit debt evolves according to the expression $d_{t+1} = (1+r)d_t - (1+r)y^T$. Since $r > 0$, we have that d_t must converge to the steady state of this expression,

$$\bar{d} \equiv \frac{y^T(1+r)}{r},$$

(the natural debt limit), for otherwise d_t will converge to infinity in absolute value at the rate r, violating the transversality condition (11). The fact that c_t^T converges to 0 implies by equation (14) that the relative price of nontradables must converge to 0, $\lim_{t\to\infty} p_t = 0$ (that is, tradables become infinitely expensive). Finally, with $p_t \to 0$ and $d_{t+1} \to y^T(1+r)/r$, in the limit the collateral constraint (4) becomes $y^T(1+r)/r \leq \kappa y^T$, which violates Assumption 1. We have therefore established that an equilibrium in which the collateral constraint never binds is impossible. We summarize this result in the following proposition:

Proposition 1 *(The recurrent nature of a binding collateral constraint). If $r > 0$, and Assumptions 1 and 2 hold, then in equilibrium the collateral constraint binds in an infinite number of periods.*

We next establish that if Assumptions 1 and 2 hold, a steady state exists, is unique, and features a binding collateral constraint. A steady state is defined as an equilibrium in which all variables are forever constant. Proposition 1 shows that an equilibrium in which the collateral constraint is always slack does not exist. We therefore have the following corollary:

Corollary 1. *If a steady state exists, it must feature a binding collateral constraint.*

To analyze existence of a steady state, it is convenient to express the right-hand side of the collateral constraint, equation (4), as a function of d_{t+1} and d_t. To this end use the resource constraint, equation (12), to eliminate c_t^T from equation (14), and then use the resulting expression to eliminate p_t from the collateral constraint. This yields

$$\kappa(y^T + p_t y^N) = F(d_{t+1}, d_t) \equiv \kappa \left[y^T + \frac{1-a}{a} \left(\frac{y^T + \frac{d_{t+1}}{1+r} - d_t}{y^N} \right)^{1/\xi} y^N \right], \tag{16}$$

with $F_1 > 0$ and $F_2 = -(1+r)F_1 < 0$. The collateral constraint (4) can then be written as

$$d_{t+1} \leq F(d_{t+1}, d_t).$$

The object $F(d_{t+1}, d_t)$ is the value of collateral in equilibrium. The intuition for why in equilibrium the value of collateral depends on the amount of debt taken on by the country in period t, d_{t+1}, is as follows. All other things equal, an increase in d_{t+1} implies an increase in consumption. Given the relative price of nontradables, p_t, households would like to spend the additional borrowing on both tradable and nontradable consumption. However, in equilibrium the supply of nontradables, y^N, is fixed. Consequently, market clearing in the nontraded sector requires an increase in the relative price of nontradables. In turn, the increase in p_t raises the value of income, $y^T + p_t y^N$, which in equilibrium is $F(d_{t+1}, d_t)/\kappa$.

From Corollary 1, we have that if a steady state exists, there must be a scalar $\tilde{d}$ such that the collateral constraint holds with equality when $d_t = d_{t+1} = \tilde{d}$. Formally, suppose that $d_t = d_{t+1} = d$, then the steady-state collateral constraint becomes

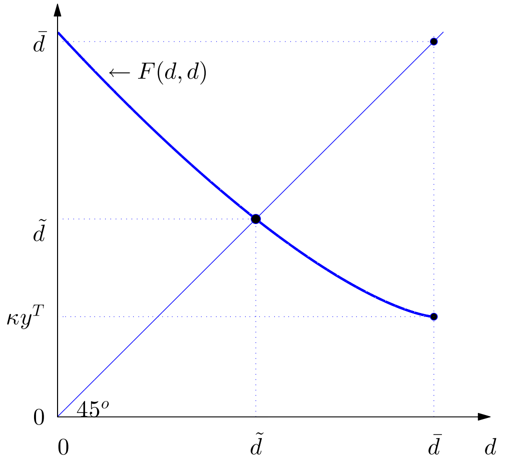
$$d \leq F(d, d) = \kappa \left[y^T + \frac{1-a}{a} \left(\frac{y^T - \frac{rd}{1+r}}{y^N} \right)^{1/\xi} y^N \right]. \tag{17}$$

Fig. 1 plots the left- and right-hand sides of the steady-state collateral constraint, equation (17). The left-hand side is the 45-degree line. The right-hand side, $F(d, d)$, is unambiguously downward sloping. The steady-state level of debt, $\tilde{d}$, is the value of d at which the left- and right-hand sides intersect, that is, where the steady state collateral constraint is binding. By Assumption 1, the collateral constraint is violated at the natural debt limit, $d = \bar{d} \equiv y^T(1+r)/r$. Also, it is clear that the collateral constraint is slack when $d = 0$. This means that $\tilde{d}$ exists, is a unique positive scalar smaller than the natural debt limit ($0 < \tilde{d} < \bar{d}$), and is implicitly given by

$$\tilde{d} = F(\tilde{d}, \tilde{d}). \tag{18}$$

Because $\tilde{d}$ is below the natural debt limit, the associated steady state value of c_t^T is strictly positive.

To complete the proof of the existence of a steady state, it remains to show that when $d_t = \tilde{d}$ for all t, the Euler equation (7), the nonnegativity constraint (9), and the transversality condition (11) all hold. The latter condition is trivially satisfied for any constant value of debt. Evaluating (7) at $\tilde{d}$ we obtain $1 = \frac{\beta(1+r)}{1-(1+r)\mu_t}$, which by Assumption 2 implies that μ_t is positive and equal to $\tilde{\mu} \equiv 1/(1+r) - \beta > 0$, so that (9) holds. Intuitively, this expression for $\tilde{\mu}$ says that the more impatient the household is (the smaller β is), the larger $\tilde{\mu}$ will be, reflecting the fact that more impatient households would be willing to pay a higher price for the right to increase their debt by one unit. We have therefore shown that all the equilibrium conditions are satisfied when $d_t = \tilde{d}$ for all t, that is, we have demonstrated the existence of a steady state. We summarize this result in the following proposition:



Notes. The figure plots the right-hand side of the steady-state collateral constraint (17) (the downward sloping line) and its left-hand side (the 45-degree line). On the horizontal axis, $\tilde{d}$ denotes the steady-state level of debt, that is, the solution to $d = F(d, d)$, and $\bar{d} \equiv y^T(1+r)/r$ denotes the natural debt limit. On the vertical axis, κy^T is the value of collateral at the natural debt limit (i.e., when $c_t^T = p_t = 0$).

Fig. 1. The steady-state collateral constraint.

Proposition 2 *(Existence of the steady state). If Assumptions 1 and 2 hold, then a steady state exists and is unique. Furthermore, the steady state features a binding collateral constraint and a level of debt implicitly given by* $\tilde{d} = F(\tilde{d}, \tilde{d})$.

4. Characterization of the debt policy function

In this section we characterize the debt policy function, which we denote by

$$d_{t+1} = D(d_t).$$

To this end we reduce the set of equilibrium conditions as follows. Using the resource constraint (12) to eliminate c_t^T from (13), we can express λ_t as the following function of d_{t+1} and d_t:

$$\lambda_t = \Lambda(d_{t+1}, d_t) \equiv U'\left(A\left(y^T + \frac{d_{t+1}}{1+r} - d_t, y^N\right)\right) A_1\left(y^T + \frac{d_{t+1}}{1+r} - d_t, y^N\right),$$

with $\Lambda_1 < 0$ and $\Lambda_2 = -(1+r)\Lambda_1 > 0$.

An equilibrium is then a pair of sequences $\{d_{t+1}, \mu_t\}_{t=0}^{\infty}$ satisfying

$$\Lambda(d_{t+1}, d_t) = \frac{\beta(1+r)}{1-(1+r)\mu_t}\Lambda(d_{t+2}, d_{t+1}), \tag{19}$$

$$d_{t+1} \leq F(d_{t+1}, d_t), \tag{20}$$

$$\mu_t[F(d_{t+1}, d_t) - d_{t+1}] = 0, \tag{21}$$

$$\mu_t \geq 0, \tag{22}$$

$$y^T + \frac{d_{t+1}}{1+r} - d_t > 0, \tag{23}$$

and

$$\lim_{t\to\infty}\frac{d_t}{(1+r)^t}=0, \tag{24}$$

given the initial level of debt, d_0. With equilibrium sequences for d_{t+1} and μ_t in hand, c_t^T can then be obtained from (12), λ_t from (13), and p_t from (14).

To avoid the type of multiplicity of equilibria identified in Schmitt-Grohé and Uribe (forthcoming), we restrict attention to parameter configurations for which the slope of the right-hand side of the collateral constraint with respect to d_{t+1} evaluated at the steady state is less than one,

Assumption 3. $F_1(\tilde{d},\tilde{d})<1$.

The interpretation of this condition is that in the vicinity of the steady state, if the country takes on more debt, the collateral constraint tightens. To see this, recall that the slack in the collateral constraint equals $F(d_{t+1},d_t)-d_{t+1}$. When Assumption 3 is not satisfied, the collateral constraint behaves perversely, in the sense that in equilibrium an increase in debt relaxes the collateral constraint. Schmitt-Grohé and Uribe (forthcoming) show that when Assumption 3 is not satisfied, multiple equilibria exist. Thus, Assumption 3 differentiates the analysis in the current paper from that in Schmitt-Grohé and Uribe (forthcoming) by showing that even if the equilibrium is unique, the economy may display endogenously generated credit cycles. Assumption 3 is neither a strong nor a weak assumption in the sense that there are equally plausible calibrations for which it is satisfied and for which it fails. As we will show below, for example, it is satisfied for the calibration adopted in Bianchi (2011), which is often used in the related literature.

For the analysis that follows it will be convenient to define the pairs (d_{t+1},d_t) such that equilibrium condition (20) holds with equality, that is, the pairs (d_{t+1},d_t) that satisfy $d_{t+1}=F(d_{t+1},d_t)$. We write this relationship as

$$G(d_t)=\{d_{t+1}:d_{t+1}=F(d_{t+1},d_t)\}. \tag{25}$$

4.1. Cobb-Douglas consumption aggregator

Before considering the case of a CES consumption aggregator, as a stepping stone, we study the special case of a Cobb-Douglas aggregator, which results under a unit intratemporal elasticity of consumption substitution,

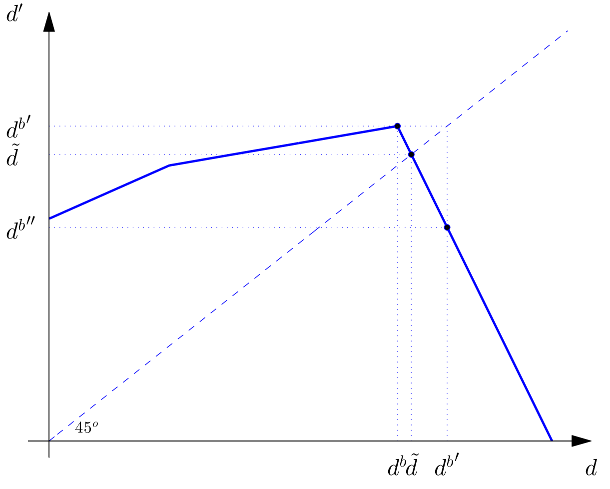
$$\xi=1.$$

Under this parameterization, the equilibrium value of collateral, $F(d_{t+1},d_t)$, becomes a linear function of debt,

$$F(d_{t+1},d_t)=\kappa y^T+\kappa\frac{1-a}{a}\left(y^T+\frac{d_{t+1}}{1+r}-d_t\right) \tag{26}$$

and the requirement that collateral increases less than one-for-one with d_{t+1}, Assumption 3, becomes

$$\frac{\kappa}{1+r}\frac{1-a}{a}<1.$$



Notes. The solid line depicts the debt policy function, $d' = D(d)$. The variable $\tilde{d}$ corresponds to the steady state and the variable d^b indicates the value of debt such that the collateral constraint is slack for $d < d^b$ and is binding for $d > d^b$.

Fig. 2. The debt policy function with a Cobb-Douglas aggregator.

When the aggregator function is Cobb-Douglas, by (25), we have that

$$G(d_t) \equiv \frac{\kappa + \kappa \frac{1-a}{a}}{1 - \frac{\kappa}{1+r}\frac{1-a}{a}} y^T - \frac{\kappa \frac{1-a}{a}}{1 - \frac{\kappa}{1+r}\frac{1-a}{a}} d_t.$$

By Assumption 3, in this case $G(\cdot)$ is downward sloping.

The main finding of this section is that the policy function, $D(\cdot)$, looks like the function depicted in Fig. 2. In particular, it is everywhere continuous and crosses the 45-degree line once and from above. Importantly, the slope of the policy function changes sign at a level of debt d^b satisfying $0 < d^b < \tilde{d}$. For levels of debt higher than d^b, the collateral constraint is binding and the policy function is the linear decreasing function $G(d_t)$. For values of debt lower than d^b, the collateral constraint is slack, and the policy function is upward sloping. Interestingly, for debt levels, d_t, in the range $[d^b, \tilde{d})$, debt increases ($d_{t+1} > d_t$) even though the collateral constraint is binding. Thus, within this range, a binding collateral constraint is not associated with deleveraging. We summarize the properties of the policy function in the following proposition:

Proposition 3 *(Properties of the policy function when $\xi = 1$). If $\xi = 1$ and Assumptions 1 to 3 hold, then the policy function, $d_{t+1} = D(d_t)$, is continuous and crosses the 45-degree line once and with negative slope at $\tilde{d}$. There exists a level of debt $d^b < \tilde{d}$ satisfying $D(d^b) > d^b$ above which the collateral constraint binds and the policy function is downward sloping and below which the collateral constraint is slack and the policy function is upward sloping.*

Proof. See Appendix A. □

The assumptions of Proposition 3 are satisfied for reasonable parameterizations, such as the one shown in Table 1 with $\xi = 1$, which, except for this parameter, is the one adopted in Bianchi (2011).

Table 1
Calibration.

Parameter	Value	Description
β	0.91	Subjective discount factor
r	0.04	Annual interest rate
κ	$0.32(1+r)$	Parameter of collateral constraint
σ	2	Inverse of intertemporal elasticity of consumption
ξ	0.83, 1	Elasticity of substitution between tradables and nontradables
a	0.31	Parameter of CES aggregator
y^T, y^N	1	Endowments

Notes. The time unit is a year. When $\xi = 0.83$, all parameter values are as in Bianchi (2011). In that study, the collateral constraint is expressed as $d_{t+1}/(1+r) = \kappa(y^T + p_t y^N)$, which means that the value of κ in this paper must be set to $(1+r)$ times the value of 0.32 in Bianchi's work.

4.2. CES consumption aggregator

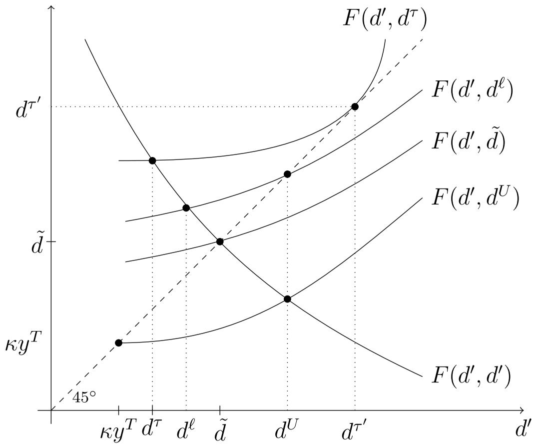
Under certain conditions to be specified here, Proposition 3 also applies when the intratemporal elasticity of substitution between tradables and nontradables is less than one,

$$0 < \xi < 1.$$

The key difference with the Cobb-Douglas case ($\xi = 1$) is that when $\xi \in (0, 1)$, the right-hand side of the collateral constraint, $F(d_{t+1}, d_t)$, ceases to be linear in d_{t+1} and becomes strictly convex. As a result, the equation $d_{t+1} = F(d_{t+1}, d_t)$ may admit two solutions for d_{t+1} given a d_t. In other words, the relation $G(d_t)$ defined in (25) may not be single valued. This feature of the model can potentially give rise to multiple equilibria of the type analyzed in Schmitt-Grohé and Uribe (forthcoming), which are not the focus of the present analysis. However, if one imposes the selection criterion of allowing only the lower value of d_{t+1} for which the collateral constraint binds,[1] then $G(d_t)$ is again single valued and decreasing. The shape of the debt policy function depends not only on the function $G(\cdot)$, but also on the composed function $G(G(\cdot))$, which specifies the evolution of debt when the collateral constraint binds in two consecutive periods. Appendices B.1 and B.2 formally characterize the functions $G(\cdot)$ and $G(G(\cdot))$. Here, we describe them in a graphical fashion.

Fig. 3 plots the right-hand side of the collateral constraint, $F(d', d)$, as a function of next-period debt, d', for three levels of current-period debt, d, which define the domains of the functions $G(\cdot)$ and $G(G(\cdot))$: d^U, d^ℓ, and d^τ. As a reference, it also plots the right-hand side of the collateral constraint for the steady-state level of debt, $\tilde{d}$. The left-hand side of the collateral constraint, d', is the 45-degree line, shown with a broken line. The figure reproduces from Fig. 1 the right-hand side of the steady-state collateral constraint, $F(d', d')$. The level of debt d^U is the upper bound of debt for which an equilibrium exists. As shown in the figure when $d = d^U$, the collateral constraint binds with a slope exactly equal to zero. The slope of the right-hand side of the collateral constraint, $F_1(d', d)$, vanishes when consumption of tradables is zero, $c^T = 0$. Any level of debt greater than d^U is unsustainable. Proposition B.1 in Appendix B establishes this result formally. The debt level d^ℓ is the level of debt such that if the collateral constraint binds, it can place the economy at d^U next period, $d^U = F(d^U, d^\ell)$. In this case, $G(G(d^\ell))$ would be associated with zero consumption of tradables. So this function would not be defined for debt

[1] This selection criterion is implicitly imposed in much of the sudden stop literature.



Notes. The figure plots the right-hand side of the collateral constraint, $F(d', d)$ as a function of next-period debt, d', for four levels of current period debt, d, namely, $\tilde{d}$, d^U, d^ℓ, and d^τ. The left-hand side of the collateral constraint is the 45-degree line. The figure reproduces from Fig. 1 the right-hand side of the long-run collateral constraint $F(d', d')$.

Fig. 3. The collateral constraint with a CES consumption aggregator, $0 < \xi < 1$.

levels less than or equal to d^ℓ. Finally, d^τ is the smallest level of debt for which the collateral constraint can bind. At this level of debt, the right-hand side of the collateral constraint meets the left-hand side only once and at a point of tangency, denoted $d^{\tau'}$, where $d^{\tau'} = F(d^{\tau'}, d^\tau)$. For levels of current debt below d^τ, the collateral constraint is slack for any choice of next-period debt, d'. So the function $G(\cdot)$ is not defined for debt levels lower than d^τ.

The following proposition states that under certain regularity conditions on the function $F(\cdot, \cdot)$, the debt policy function for the economy with a CES consumption aggregator has the same characteristics as that associated with the economy with a Cobb Douglas consumption aggregator (see Fig. 2).

Proposition 4 *(Properties of the policy function with a CES aggregator). If $0 < \xi < 1$, and Assumptions 1 to 3 and B.1 (given in Appendix B) hold, then under the equilibrium selection criterion of allowing only the lower value of debt for which the collateral constraint is binding, the policy function, $d_{t+1} = D(d_t)$, is continuous and crosses the 45-degree line once and with negative slope at $\tilde{d}$. There exists a level of debt $d^b < \tilde{d}$ satisfying $D(d^b) > d^b$ above which the collateral constraint binds and the policy function is downward sloping and below which the collateral constraint is slack and the policy function is upward sloping.*

Proof. See Appendix B. □

The assumptions of Proposition 4 are satisfied for the calibration shown in Table 1 with $\xi = 0.83$, which is the calibration assumed in Bianchi (2011).

5. Endogenous debt cycles

The present economy can exhibit bounded equilibrium dynamics that do not converge to the steady state. Because the parameterizations we focus on yield a single-valued policy function, the equilibrium is unique. This means that the equilibrium dynamics in this class of open economy models are inherently cyclical even in the absence of (fundamental or nonfundamental) uncertainty.

5.1. Stability of the steady state $\tilde{d}$

Consider first the equilibrium dynamics in the vicinity of the steady state, $\tilde{d}$. We have shown that for $d_t > d^b$, the policy function is given by $d_{t+1} = G(d_t)$. Because $d^b < \tilde{d}$, we have that the local stability of the steady state is determined by $G'(\tilde{d})$. The steady state is stable if $|G'(\tilde{d})| < 1$ and is unstable otherwise. Recalling that the function $G(\cdot)$ is defined as the solution of $d_{t+1} = F(d_{t+1}, d_t)$ for d_{t+1} given d_t, we have that

$$G'(\tilde{d}) = -\frac{(1+r)F_1(\tilde{d},\tilde{d})}{1-F_1(\tilde{d},\tilde{d})}.$$

The fact that F_1 is positive together with Assumption 3 guarantees that $G'(\tilde{d})$ is negative. This means that the steady state is locally stable if and only if $F_1(\tilde{d},\tilde{d}) < 1/(2+r)$. This condition says that local stability of the steady state requires that for each unit increase in debt the value of collateral increase by less than one half. We summarize this result in the following proposition:

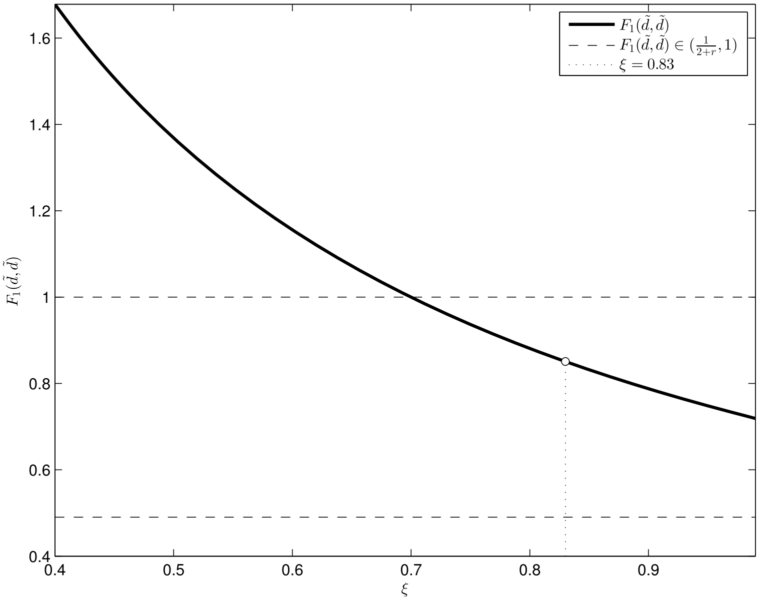
Proposition 5 *(Local stability of the steady state). Suppose that Assumptions 1 to 3 hold. Then, the steady state is locally stable if and only if $F_1(\tilde{d},\tilde{d}) < 1/(2+r)$.*

If the stability condition of Proposition 5 is not met, then a small deviation of d from its steady-state value $\tilde{d}$ will trigger dynamics leading away from and never converging back to $\tilde{d}$.

Fig. 4 shows that this is indeed the case for parameterizations commonly used in the sudden stop literature. It plots $F_1(\tilde{d},\tilde{d})$ as a function of the intratemporal elasticity of substitution, ξ. All other parameters of the model take the values shown in Table 1. For values of ξ larger than 0.7, $F_1(\tilde{d},\tilde{d})$ lies in the interval $(1/(2+r), 1)$ and therefore the steady state is locally unstable. This is the case, in particular, for $\xi = 0.83$. As we will see shortly, in this parameter region, the equilibrium is unique but the economy displays endogenous cycles. For values of ξ less than 0.7, $F_1(\tilde{d},\tilde{d}) > 1$ and therefore Assumption 3 is violated. In this parameter region, as shown in Schmitt-Grohé and Uribe (forthcoming), the economy displays multiple equilibria.

5.2. Limit cycles

A natural question is how debt behaves globally when the steady state is unstable. As it turns out, the model possesses attracting forces that prevent debt from exploding. Specifically, if the steady state is unstable, then the equilibrium exhibits bounded oscillating dynamics, which never converge to the steady state. To see this, note that, given an arbitrary initial debt level d_0: (a) if the steady state is unstable, the economy will not converge to it; and, from Propositions 3 and 4 (with graphical representation in Fig. 2); (b) when $d_t < \tilde{d}$, then $d_{t+1} > d_t$; (c) if $d_t < \tilde{d}$, then there is a finite J such that $d_{t+J} \geq d^b$; (d) if $d_t \in (d^b, \tilde{d})$, then $d_{t+1} > \tilde{d}$; and finally (e) if $d_t > \tilde{d}$, then $d_{t+1} < \tilde{d}$. Thus, the economy fluctuates perpetually around the steady state $\tilde{d}$ without ever converging to it or exploding. This type of dynamics arises because the steady state is locally repellent but globally attracting. Therefore, the equilibrium consists of an infinite



Notes. Values of ξ for which $F_1(\tilde{d}, \tilde{d}) \in (1/(2+r), 1)$ are associated with an unstable steady state. The bullet indicates the value of $F_1(\tilde{d}, \tilde{d})$ at $\xi = 0.83$. All other parameters of the function $F(\cdot, \cdot)$ are set at the values shown in Table 1.

Fig. 4. Stability condition and the intratemporal elasticity of substitution.

sequence of episodes in which debt expansions (credit booms) are followed by debt contractions (macroeconomic deleveraging). We then have the following proposition:

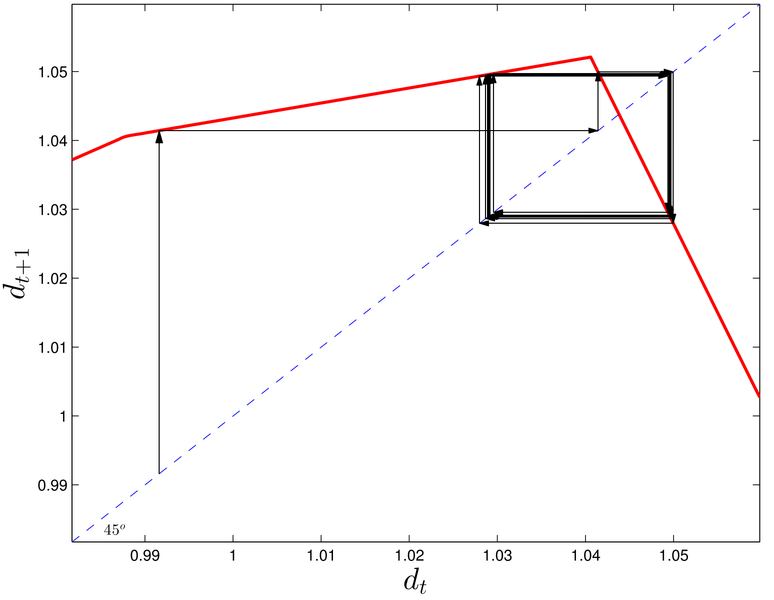
Proposition 6 *(Endogenous debt cycles). Suppose that $F_1(\tilde{d}, \tilde{d}) \in (1/(2+r), 1)$ and that $\xi = 1$ and the conditions of Proposition 3 are satisfied or that $\xi \in (0, 1)$ and the conditions of Proposition 4 are satisfied. Then, the equilibrium exhibits bounded oscillating dynamics in which debt perpetually fluctuates around its steady state $\tilde{d}$ without ever converging to it.*

5.3. Two-period cycles

Figs. 5 and 6 plot the equilibrium path of debt for an arbitrary initial condition in two calibrated economies. In Fig. 5 all parameters take the values shown in Table 1 with $\xi = 1$ (Cobb-Douglas aggregator). In the economy depicted in Fig. 6, all parameters are set at the values shown in Table 1 with $\xi = 0.83$ (CES aggregator).

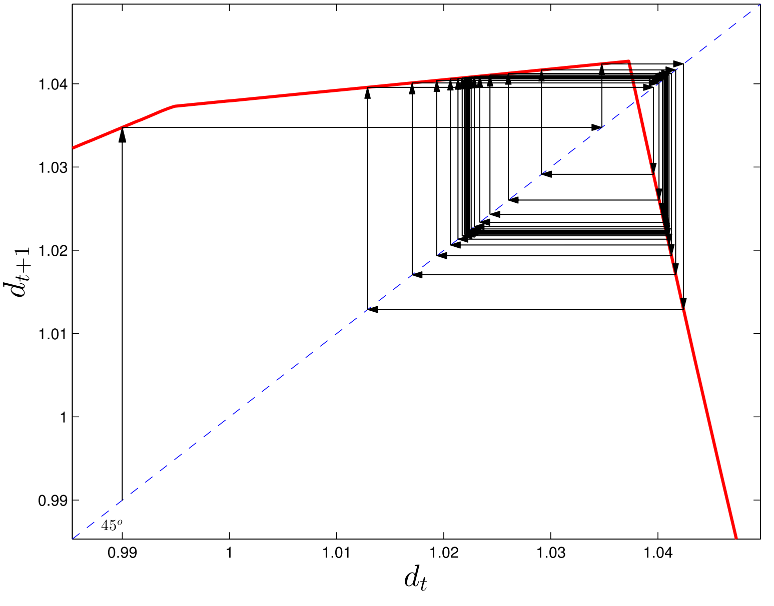
Under both parameter configurations debt converges to a two-period cycle. In the limit cycle, periods of slack collateral constraints coincide with periods of debt growth (credit booms) and periods of binding collateral constraints coincide with debt deleveraging. During a credit boom, consumption of tradables expands, equation (12), and the real exchange rate appreciates, equation (14). The opposite happens when the economy deleverages, namely, domestic absorption falls and the real exchange rate depreciates.

It is of interest to ascertain how volatile common indicators of the business cycle, such as consumption or the trade balance, are in the model. At the calibration shown in Fig. 6, which targets the Argentine economy (see Bianchi, 2011, for details), the standard deviation of consumption implied by the 2-period cycle is 2.2 percent. For the Argentine data, Bianchi (2011) reports a



Note. The structural parameters take the values shown in Table 1 with $\xi = 1$.

Fig. 5. Convergence to a two-period cycle: Cobb-Douglas aggregator.



Note. The structural parameters take the values shown in Table 1 with $\xi = 0.83$.

Fig. 6. Convergence to a two-period cycle: CES aggregator.

standard deviation of consumption of 6.2 percent. Further, the implied standard deviation of the trade-balance-to-output ratio is 0.6 percentage points, compared to an observed standard deviation of 2.4 percentage points. Thus, the model predicts standard deviations for consumption and the trade balance that are one half and one fourth of their empirical counterparts, respectively. We interpret this result as suggesting that the amplitude of the business cycle associated with the periodic equilibrium is not negligible. The amplitude of the endogenous credit cycle is most sensitive to the value of the subjective discount factor, β. In the model impatient households negotiate a tradeoff between front-loading consumption by taking on more debt in the short run than is sustainable in the long and having to deleverage in the future. The more impatient households are, the larger their attempts to frontload consumption and the larger the necessity to deleverage will be. In the open economy literature with financial constraints, households are typically calibrated to be highly impatient. The reason is that the more impatient households are, the more likely it is that the financial constraints will be binding frequently in equilibrium. For example, Arellano (2008) and Chatterjee and Eyigungor (2012) in the context of models with imperfect enforcement of international debt contracts assume a value of β of 0.953 at a quarterly frequency, which is equivalent to setting this parameter at 0.825 at an annual frequency. Setting β to 0.825, while leaving all other parameters at their values given in Table 1, implies the existence of a two-period cycle in which the standard deviation of consumption and the trade-balance-to-output ratio are 6 percent and 1.7 percentage points, respectively. These values are fairly close to their empirical counterparts.

Let's now explore more formally the existence and stability of two-period cycles. Consider a two-period cycle in which the collateral constraint binds every other period, which, for example, is the case for the two-period cycles shown in Figs. 5 and 6. Let d^c and d^u be the levels of debt in periods in which the economy is constrained and unconstrained, respectively. In a period in which the constraint is slack, the Lagrange multiplier on the collateral constraint is nil ($\mu = 0$), so that the equilibrium Euler equation (19) takes the form

$$\Lambda(d^u, d^c) = \beta(1+r)\Lambda(d^c, d^u).$$

When the economy is constrained, the next-period debt satisfies

$$d^c = G(d^u).$$

The above two equations uniquely determine d^u and d^c. If in addition the collateral constraint is satisfied in the period in which the economy is unconstrained, $d^u \leq F(d^u, d^c)$, and if consumption is positive in both states, $y^T + d_{t+1}/(1+r) - d_t > 0$ for $(d_{t+1}, d_t) = (d^c, d^u)$, (d^u, d^c), then a two-period cycle exists, with periodic points d^u and d^c. When the aggregator function is Cobb-Douglass ($\xi = 1$), both of the above equations are linear, which allows for a closed-form solution of the cycle. When the aggregator function is of the CES form ($\xi \in (0, 1)$), the two-period cycle can be computed using numerical methods.

Consider now the stability of the two-period cycle. Does the economy converge to the cycle (d^c, d^u) for arbitrary initial debt levels in the vicinity of d^c or d^u? Suppose that the economy is sufficiently close to the limit cycle, so that it continues to be the case that it is constrained every other period. Let d_t be a period in which the collateral constraint is slack. Then, the period-t Euler equation holds with $\mu_t = 0$, and the period $t+1$ collateral constraint is binding, $d_{t+2} = G(d_{t+1})$. So we have that the Euler equation in period t can be written as

$$\Lambda(G^{-1}(d_{t+2}), d_t) = \beta(1+r)\Lambda(d_{t+2}, G^{-1}(d_{t+2})), \tag{27}$$

where $G^{-1}(\cdot)$ is the inverse function of $G(\cdot)$, so that $d_{t+1} = G^{-1}(d_{t+2})$. Equation (27) is a first-order difference equation defining the law of motion of debt chosen in periods in which the collateral constraint is binding, which can be written as

$$d_{t+2} = C_2(d_t),$$

where the subscript indicates the periodicity of the cycle. The steady state of this equation is d^c. The two-period cycle is locally stable (attracting) if $|C_2'(d^c)| < 1$, provided $C_2(\cdot)$ is differentiable in the vicinity of d^c. In this case, full differentiation of (27) evaluated at $d_t = d_{t+2} = d^c$ yields

$$C_2'(d^c) = \frac{(1+r)G'(d^u)}{1 - \beta(1+r)\frac{\Lambda_1(d^c,d^u)}{\Lambda_1(d^u,d^c)}[G'(d^u) - (1+r)]} < 0,$$

where $d^u = G^{-1}(d^c)$. The fact that C_2' is negative means that if the two-period cycle is attracting, the convergence to it is oscillatory, as can be observed in Figs. 5 and 6.

The stability of the two-period cycle, however, cannot be determined by simple inspection of this expression. Although a closed-form solution for $C_2'(d^c)$ can be obtained when $\xi = 1$ (Cobb-Douglas aggregator), it is an involved function of the structural parameters of the model. One must therefore resort to a numerical evaluation of $C_2'(d^c)$. We note that this is an exact numerical evaluation. When $\xi < 1$ (CES aggregator), $C_2'(d^c)$ can be approximated numerically to any degree of accuracy. Setting $\xi = 1$ and setting all other parameters at their values given in Table 1 yields $C_2'(d^c) = -0.56$, confirming that the two-period cycle shown in Fig. 5 is stable. Similarly, setting $\xi = 0.83$ yields $C_2'(d^c) = -0.74$, which says that the two-period cycle shown in Fig. 6 is also attracting. We conclude that the economy exhibits an attracting two-period cycle for parameterizations commonly used in the sudden stop literature.

5.4. Three-period cycles

Suppose that for a given parameterization both the steady state and the two-period cycle are unstable ($G'(\tilde{d}), C_2'(d^c) < -1$). What do the equilibrium dynamics look like in this case? We have already established that when $G'(\tilde{d}) < -1$, the equilibrium level of debt fluctuates perpetually around $\tilde{d}$ without converging to it (Proposition 6).

Here, we show that there exist parameterizations for which the equilibrium dynamics exhibit three-period cycles. This type of periodic equilibria comes in only two forms: one featuring two consecutive periods with a slack collateral constraint followed by a period with a binding constraint, and the other featuring a period with a slack collateral constraint followed by a period with a binding constraint in which nonetheless credit expands, followed by another period with a binding constraint, in which the economy is forced to deleverage.

Three-period cycles in which the collateral constraint is always binding are impossible. To see this, suppose that such a cycle exists. Let the periodic debt levels be denoted d^c, d^{cc}, and d^{ccc}. Suppose, without loss of generality, that $d^c \in (d^b, \tilde{d})$, and that it is followed by d^{cc}. Then, since $\tilde{d} = G(\tilde{d})$ and $G'(\cdot) < 0$, we have that $d^{cc} = G(d^c) > \tilde{d}$, $d^{ccc} = G(d^{cc}) < \tilde{d}$, and $d^c = G(d^{ccc}) > \tilde{d}$, which is a contradiction. Similarly, three-period cycles in which the economy is always unconstrained are impossible by Proposition 1.

Consider the first type of three-period cycle, namely, the one in which the economy is unconstrained in two periods and constrained in the third.[2] Let d^u be the level of debt chosen when

[2] The characterization of the second type of period-3 cycles goes along similar lines, and we therefore omit it.

the economy is financially unconstrained in the current and the next period, and let d^{uu} be the level of debt chosen when the economy is unconstrained in the current period but constrained in the next. And let d^c be the level of debt chosen in periods in which the economy is financially constrained. When the chosen level of debt is d^u or d^{uu}, the multiplier μ is nil, and the Euler equations take the form, respectively,

$$\Lambda(d^u, d^c) = \beta(1+r)\Lambda(d^{uu}, d^u),$$

and

$$\Lambda(d^{uu}, d^u) = \beta(1+r)\Lambda(d^c, d^{uu}).$$

When the collateral constraint is binding, next-period debt satisfies

$$d^c = G(d^{uu}). \tag{28}$$

The above expressions form a system of three equations in three unknowns, which can be solved for d^u, d^{uu}, and d^c. If the collateral constraint is not violated at either of the two periodic points in which the economy is unconstrained, $d^u \leq F(d^u, d^c)$ and $d^{uu} \leq F(d^{uu}, d^u)$, and if consumption of tradables is positive at the three periodic points, $y^T + d_{t+1}/(1+r) - d_t > 0$ for $(d_{t+1}, d_t) = (d^{uu}, d^u)$, (d^u, d^c), (d^c, d^{uu}), then the triplet (d^c, d^u, d^{uu}) represents a three-period cycle.

Three-period cycles of this type can be found for parameterizations close to the one given in Table 1. To facilitate computations, we focus on the Cobb-Douglas case ($\xi = 1$) as it admits a closed-form solution. Three-period cycles do not exist when all parameters take the values shown in Table 1 with $\xi = 1$. But plausible calibrations do exhibit this type of endogenous fluctuations. As an example, consider the parameter configuration $a = 0.23$, $\beta = 0.88$, $\kappa = 0.32$, $r = 0.091$, and $\sigma = 1.7$. Fig. 7 displays the path of debt along the three-period cycle associated with this parameterization. The elevated segments of the time path are the periods in which the economy is unconstrained, $d_{t+1} = d^u, d^{uu}$. The troughs correspond to periods in which the collateral constraint binds and the economy deleverages, $d_{t+1} = d^c$. The fact that the economy has a three-period cycle implies, as we will discuss shortly, that it must also have a two-period cycle, which is shown in Fig. 7 with a broken line. For the particular parameterization considered, the amplitude of the three-period cycle is larger than that of the two-period cycle.

The three-period cycle displayed in Fig. 7 is stable, while, by construction, the two-period cycle and the steady state (the one-period cycle) are unstable (recall that we are focusing attention on parameter values for which this is the case). The stability of the three-period cycle can be verified by simulation, as shown in Fig. 8, or analytically following the same steps as in the stability analysis of the two-period cycle. For the latter approach, write the equilibrium law of motion of debt near the three-period cycle as

$$\Lambda(d_{t+1}, d_t) = \beta(1+r)\Lambda(G^{-1}(d_{t+3}), d_{t+1}) \tag{29}$$

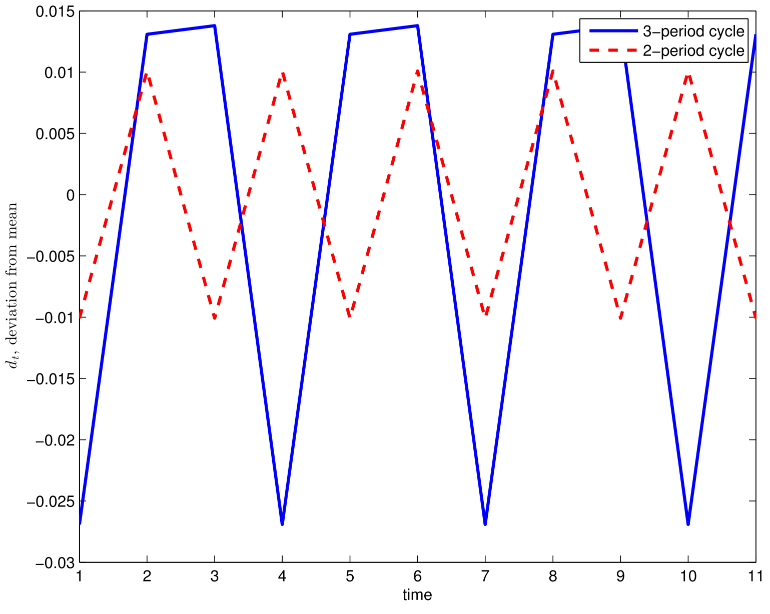
and

$$\Lambda(G^{-1}(d_{t+3}), d_{t+1}) = \beta(1+r)\Lambda(d_{t+3}, G^{-1}(d_{t+3})). \tag{30}$$

Combining these two expressions to eliminate d_{t+1} yields an implicit function describing the evolution of debt in periods in which the collateral constraint binds

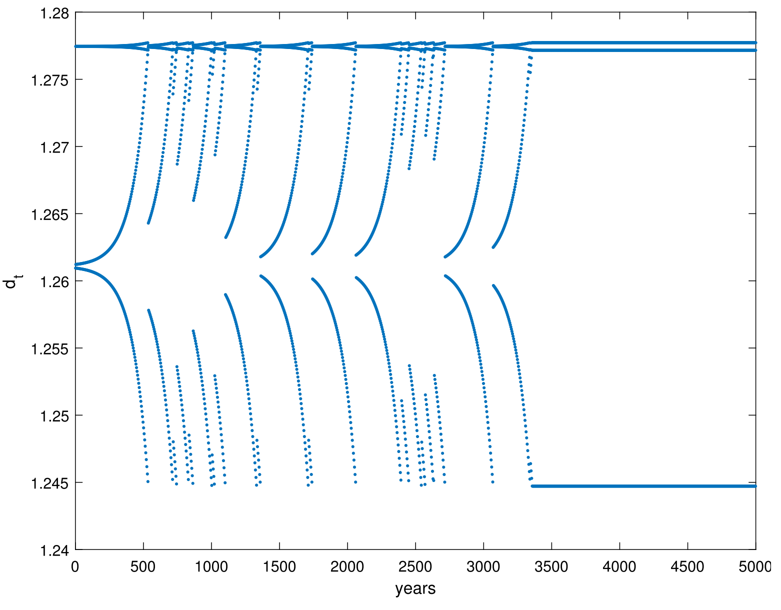
$$d_{t+3} = C_3(d_t).$$

The steady state of this difference equation is d^c, that is, $d^c = C_3(d^c)$. Local stability of the three-period cycle requires that $|C_3'(d^c)| < 1$, provided that $C_3(\cdot)$ is differentiable at d^c. If this



Notes. The structural parameters take the following values: $a = 0.23$, $\beta = 0.88$, $\kappa = 0.32$, $r = 0.091$, $\sigma = 1.7$, $\xi = 1$, and $y^T = y^N = 1$. Debt is expressed in deviations from its mean value.

Fig. 7. Three-period cycles.



Notes. The structural parameters take the following values: $a = 0.23$, $\beta = 0.88$, $\kappa = 0.32$, $r = 0.091$, $\sigma = 1.7$, $\xi = 1$, and $y^T = y^N = 1$.

Fig. 8. Convergence to a three-period cycle: Cobb-Douglas aggregator.

is the case, then fully differentiating (29) and (30) and evaluating the derivatives at $d_t = d^c$, $d_{t+1} = d^u$, $d_{t+2} = d^{uu}$, and $d_{t+3} = d^c$, we have

$$C_3'(d^c) = \frac{(1+r)A}{1 - \beta(1+r)\frac{\Lambda_1(d^{uu},d^u)}{\Lambda_1(d^u,d^c)}(1+r)\left[\frac{A}{(1+r)G'(d^{uu})} - 1\right]} < 0,$$

where

$$A \equiv \frac{(1+r)G'(d^{uu})}{1 - \beta(1+r)\frac{\Lambda_1(d^c,d^{uu})}{\Lambda_1(d^{uu},d^u)}[G'(d^{uu}) - (1+r)]} < 0.$$

The inequality follows from the fact that $\frac{A}{(1+r)G'(d^{uu})} < 1$ and implies that, if the three-period cycle is attracting, then the convergence toward it is oscillatory. Evaluating the above expression at the periodic values of debt (d^c, d^u, and d^{uu}), we obtain that $C_3'(d^c) = -0.53$, confirming the attracting nature of the cycle.

5.5. *Cycles of any periodicity and chaos*

In a seminal contribution, Li and Yorke (1975) show that if a univariate difference equation has a cycle of periodicity three, then it has cycles of any periodicity and chaos. Chaotic dynamics are dynamics in which debt does not converge asymptotically to a cycle of any periodicity (including a unit periodicity, the steady state). Since as shown above, three-period cycles exist for plausible parameterizations, we have the following proposition:

Proposition 7. *If the conditions of Proposition 3 hold, then there exist plausible parameterizations for which the economy displays cycles of any periodicity and chaos.*

Proof. See Appendix C. □

The theorem of Li and Yorke, however, does not indicate the measure of the set of initial debt levels that give rise to chaotic dynamics. Indeed, such set may be of measure zero. For the economy studied in this paper, we could not detect, using numerical methods, parameters for which the equilibrium dynamics converge to cycles with periodicity higher than three. However, as shown in Fig. 8, the transitional dynamics converging to a three-period cycle can look quite complex for long periods of time. Specifically, in the figure, a clear convergence pattern is discernible only after more than 3,000 years, even though the economy starts from a point near the cycle.

6. Optimal policy

Would a benevolent government wish to eliminate the endogenous cycles that inevitably occur under laissez-faire? To address this question, we consider the constrained optimal allocation, defined as the solution to the problem of a benevolent social planner who faces the collateral constraint and internalizes that in equilibrium the relative price of nontradables—and thereby the value of collateral—depends on aggregate absorption and that the market for nontradables must clear. We first show that the constrained optimal allocation does not display endogenous cycles and then describe a fiscal policy that supports the desired equilibrium.

6.1. Ramsey optimal policy

The optimization problem of the Ramsey planner is to choose sequences $\{c_t^T, d_{t+1}\}_{t=0}^{\infty}$ to maximize

$$\sum_{t=0}^{\infty} \beta^t U(A(c_t^T, y^N))$$

subject to

$$c_t^T + d_t = y^T + \frac{d_{t+1}}{1+r}, \tag{31}$$

$$d_{t+1} \leq H(c_t^T) \equiv \kappa \left[y^T + \left(\frac{1-a}{a} \right) c_t^{T\,\frac{1}{\xi}} y^{N\,1-\frac{1}{\xi}} \right], \tag{32}$$

with $c_t^T > 0$ and $\lim_{t\to\infty} d_t(1+r)^{-t} \leq 0$. Let λ_t^R and $\lambda_t^R \mu_t^R$ denote the Lagrange multipliers on the resource constraint (31) and the collateral constraint (32), respectively. The optimality conditions associated with this problem are

$$U'(A(c_t^T, y^N)) A_1(c_t^T, y^N) = \lambda_t^R \left[1 - H'(c_t^T) \mu_t^R \right]$$

and

$$\left(1 - (1+r)\mu_t^R \right) \lambda_t^R = \beta(1+r) \lambda_{t+1}^R.$$

The planner's Euler equation is identical to that of the individual household. However, her marginal utility of wealth, λ_t^R, is different, as she internalizes that a unit increase in consumption of tradables has a positive shadow value when the collateral constraint is binding stemming from its positive effect on the value of collateral via the boosting of the relative price of nontradables, $H'(c_t^T) > 0$.

Using the resource constraint to eliminate c_t^T from the collateral constraint, noting that $H'(c_t^T) = (1+r)F_1(d_{t+1}, d_t)$ and letting, as before, $\Lambda(d_{t+1}, d_t) = U'(A(c_t^T, y^N)) A_1(c_t^T, y^N)$, we have that the constrained optimal allocation are sequences $\{c_t^T, d_{t+1}, \mu_t^R\}_{t=0}^{\infty}$ satisfying

$$\frac{\Lambda(d_{t+1}, d_t)}{1 - (1+r)F_1(d_{t+1}, d_t)\mu_t^R} \left[1 - (1+r)\mu_t^R \right] = \beta(1+r) \frac{\Lambda(d_{t+2}, d_{t+1})}{1 - (1+r)F_1(d_{t+2}, d_{t+1})\mu_{t+1}^R} \tag{33}$$

$$d_{t+1} \leq F(d_{t+1}, d_t) \tag{34}$$

$$\mu_t^R \geq 0, \tag{35}$$

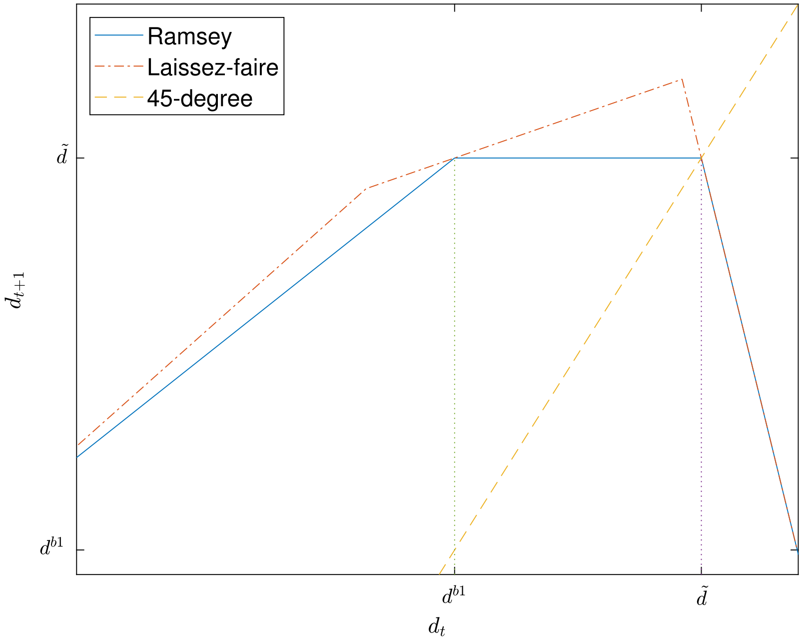
$$\mu_t^R \left[d_{t+1} - F(d_{t+1}, d_t) \right] = 0, \tag{36}$$

$$y^T + \frac{d_{t+1}}{1+r} - d_t > 0, \tag{37}$$

and

$$\lim_{t\to\infty} \frac{d_t}{(1+r)^t} = 0. \tag{38}$$

It can readily be established that Propositions 1 and 2 hold. That is, in the Ramsey optimal equilibrium the collateral constraint binds in an infinite number of periods. And this implies that



Note. The figure is drawn for the calibration shown in Table 1 with $\xi = 1$ (Cobb-Douglas aggregator). The variable $\tilde{d}$ corresponds to the steady state level of debt. Under the Ramsey policy the collateral constraint is slack for $d_t < \tilde{d}$ and binding otherwise.

Fig. 9. The Ramsey optimal debt policy function.

in the steady state the collateral constraint binds and debt is given by $\tilde{d}$. Thus, the steady state is the same in the Ramsey and unregulated equilibria. We collect this result in the following proposition:

Proposition 8 *(Steady state of the Ramsey economy). If Assumptions 1 to 3 hold, then in the Ramsey equilibrium the collateral constraint binds in an infinite number of periods. A steady state exists and is unique. Further, the steady state features a binding collateral constraint and a level of debt implicitly given by $\tilde{d} = F(\tilde{d}, \tilde{d})$.*

The Ramsey planner finds it optimal to eliminate the endogenous cycles that exist under laissez-faire. Fig. 9 plots the debt policy function, $d_{t+1} = D^r(d_t)$, associated with the constrained optimal allocation for the calibration shown in Table 1 with $\xi = 1$ (Cobb-Douglas aggregator). For comparison, it reproduces from Fig. 5 the debt policy function under laissez-faire. Under the Ramsey optimal policy, for levels of debt below the steady state ($d_t < \tilde{d}$), debt converges to the steady state monotonically and in finite time. Along the transition the collateral constraint is slack. Once debt exceeds the threshold d^{b1} (to be characterized in Proposition 9 below), the policy function becomes constant and equal to $\tilde{d}$, implying that for any level of debt between d^{b1} and $\tilde{d}$ the economy reaches the steady state $\tilde{d}$ in one period. This means that if the initial level of debt is below $\tilde{d}$, then the economy does not suffer a binding collateral constraint followed by deleveraging anywhere along the transition path. For initial levels of debt above $\tilde{d}$, the economy deleverages for one period to a value of debt below the steady state and then converges monotonically to the steady-state $\tilde{d}$ in finite time from below.

The absence of cycles under the Ramsey policy is not limited to the calibration shown in Table 1. The planner is also able to eliminate cycles, for example, for the calibration considered in section 5.4, which delivers a three-period cycle under laissez-faire. The following proposition provides conditions under which cycles are impossible in the Ramsey equilibrium.

Proposition 9 *(No endogenous cycles or deleveraging under Ramsey optimal policy). Suppose* $\xi = 1$*, Assumptions 1 to 3 hold, and* $F_1(\tilde{d}, \tilde{d}) > 1/[1 + \beta(1 + r)]$*. Then there exists an integer* $i \geq 0$ *such that the Ramsey equilibrium path of debt is of the form* $(d_t, d_{t+1}, \ldots, d_{t+i}, \tilde{d}, \tilde{d}, \ldots)$*. If* $d_t < \tilde{d}$*, the equilibrium path satisfies* $d_t < d_{t+1} < \cdots < d_{t+i} < \tilde{d}$*. And if* $d_t > \tilde{d}$*, it satisfies* $d_{t+1} < d_{t+2} < \cdots < d_{t+i} < \tilde{d} < d_t$*.*

Proof. See Appendix D. □

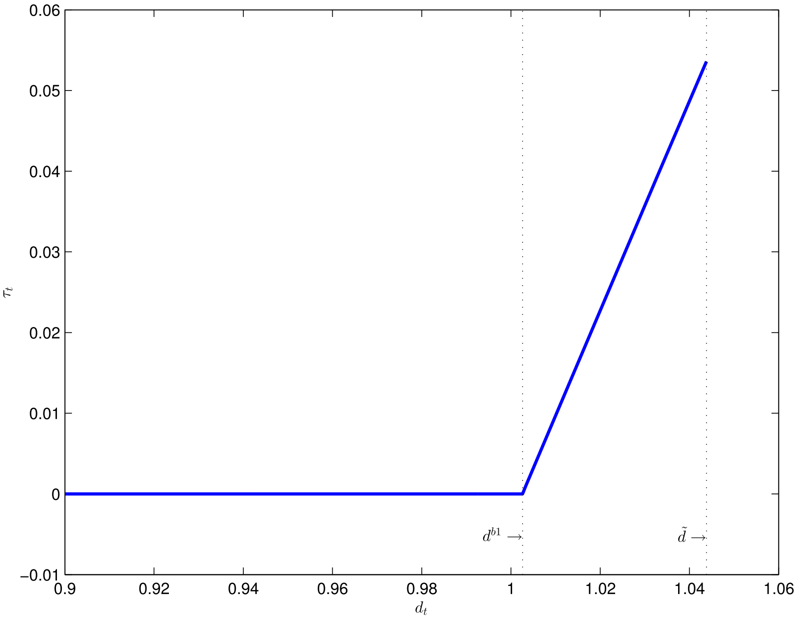
The maximum level of debt chosen by the planner is $\tilde{d}$, that is, $D^r(d_t) \leq \tilde{d}$, for all d_t. By contrast, as we established in section 5, in the laissez-faire equilibrium, the economy converges to a cycle in which periodically agents choose levels of debt exceeding $\tilde{d}$, which subsequently force them into deleveraging. When the economy is still unconstrained but close to a binding collateral constraint, the planner avoids borrowing beyond $\tilde{d}$ and therefore a future deleveraging crisis by becoming effectively more patient than agents in the laissez-faire economy. To see this, consider the Euler equation of the planner in a period in which the collateral constraint is slack ($\mu_t^R = 0$) but binds in the following period ($\mu_{t+1}^R > 0$):

$$\Lambda(d_{t+1}, d_t) = \beta_{t+1}^R (1 + r) \Lambda(d_{t+2}, d_{t+1}),$$

where $\beta_{t+1}^R \equiv \frac{\beta}{1-(1+r)F_1\mu_{t+1}^R} > \beta$. This situation occurs for any $d_t \in [d^{b1}, \tilde{d})$. In the laissez-faire equilibrium the effective discount factor is time invariant and equal to β. The decrease in the effective discount factor of the planner makes her put greater value on the costs of a future deleveraging while leaving the benefits of current spending unchanged. Therefore, she resolves the tradeoff between curbing spending today and deleveraging tomorrow in favor of the former. By contrast, private agents faced with the same tradeoff but a lower discount factor ($\beta < \beta_{t+1}^R$) choose not to curb spending today and to deleverage in the future.

If the condition $F_1(\tilde{d}, \tilde{d}) > 1/[1 + \beta(1 + r)]$ in Proposition 9 is not met, then the Ramsey optimal debt policy function ceases to be flat to the left of $\tilde{d}$. Instead, like its counterpart in the unregulated economy, it peaks before $\tilde{d}$ and the collateral constraint binds, i.e., $d_{t+1} = G(d_t)$, to the right of this peak. Under such parameterizations, the planner is not able to avoid that for some initial debt levels the economy borrows beyond the maximum long-run sustainable debt level $\tilde{d}$ and then suffers a binding constraint and debt deleveraging.

One way to gauge the difference between the allocation under the Ramsey optimal policy and under laissez-faire is to compute the welfare costs of the latter relative to the former. Consider the calibration shown in Table 1 with $\xi = 0.83$. As shown earlier, for this parameterization the laissez-faire economy displays an attracting 2-period cycle, whereas the Ramsey economy reaches the steady state, $\tilde{d}$, in finite time. The average welfare cost of living in the cyclical equilibrium is 0.02 percent of consumption each period. This number is small as is often the case in models of the business cycle including models of sudden stops. For example, in a version of the present model under the same calibration but driven by fundamental shocks calibrated to match the volatility of tradable and nontradable output in Argentina—arguably one of the most volatile



Note. The figure is drawn for the calibration shown in Table 1 with $\xi = 1$ (Cobb-Douglas aggregator).

Fig. 10. The optimal capital control policy.

emerging countries in the world,—Bianchi (2011) finds an average welfare cost of 0.135 percent of consumption.

We note, nonetheless, that like the amplitude of the endogenous cycle, the welfare cost of the periodic equilibrium is sensitive to the degree of impatience displayed by agents. For the level of impatience used in Arellano (2008) and Chatterjee and Eyigungor (2012) ($\beta = 0.825$), the average welfare cost of living in the economy with the endogenous cycle relative to living in the Ramsey economy is 0.11 percent of consumption, which is more than five times larger than when β takes its baseline value of 0.91. For very high levels of impatience, the welfare gains of internalizing the pecuniary externality can be substantial. For example, setting $\beta = 0.6$ yields an average welfare gain of 0.67 percent of consumption.

6.2. *Optimal capital control policy*

We have established that left to its own devices the economy displays a unique equilibrium characterized by endogenous debt cycles and that a benevolent government finds such cycles undesirable. As is well known, in environments like the one studied here, the Ramsey optimal allocation can be supported by a capital control tax (see, for example, Korinek, 2011; Bianchi, 2011; Benigno et al., 2013 and 2016; Bianchi and Mendoza, 2018; Dávila and Korinek, 2018; Jeanne and Korinek, 2019; and the survey by Rebucci and Ma, 2019). Specifically, suppose the government imposes capital controls that take the form of a proportional tax on debt at the rate τ_t. In this case the household's budget constraint becomes

$$c_t^T + p_t c_t^N + d_t = y^T + p_t y^N + \frac{1-\tau_t}{1+r} d_{t+1} + T_t, \tag{39}$$

where T_t denotes lump-sum transfers, which the government uses to rebate any revenues from the capital control tax. The equilibrium Euler equation becomes

$$[(1-\tau_t)-(1+r)\mu_t]\,\Lambda(d_{t+1},d_t)=\beta(1+r)\Lambda(d_{t+2},d_{t+1}). \tag{40}$$

For any $d_t < \tilde{d}$, the Ramsey optimal capital control tax results from evaluating this expression at $\mu_t = 0$, $d_{t+1} = D^r(d_t)$ and $d_{t+2} = D^{r2}(d_t)$, and solving for τ_t. Therefore having obtained the Ramsey optimal debt policy, solving for the optimal tax rate is straightforward.

Fig. 10 plots the Ramsey optimal tax rate, τ_t, for the calibration shown in Table 1 with $\xi = 1$. For $d_t \leq d^{b1}$ the tax rate is zero and the Ramsey planner's collateral constraint is slack in periods t and $t+1$. For this reason the planner lets capital flow unfettered. The planner begins to impose capital controls when the collateral constraint is slack in period t but binding in $t+1$, that is, for $d_t \in (d^{b1}, \tilde{d})$. The closer d_t is to $\tilde{d}$, the higher the tax rate will be. The tax rate reaches 5.4 percent at its peak, which means that the effective annual interest rate charged to domestic households, given by $(1+r)/(1-\tau_t)-1$, reaches 9.9 percent.

As stressed in the related literature, the purpose of the optimal capital control tax is to make households internalize that their collective absorption, by appreciating the real exchange rate, elevates the value of collateral. The novel insight of the present analysis is that government intervention is called for even in the absence of fundamental uncertainty. The reason is that the equilibrium in the laissez-faire economy features inefficient oscillations around the highest sustainable level of debt, $\tilde{d}$. These oscillations can be periodic (if endogenous cycles exist) or dampening. But they always imply inefficient credit booms followed by costly Fisherian deflations.

7. Conclusion

In much of the open economy literature, financial constraints play the role of amplifying the effects of exogenous shocks. Credit cycles are driven by exogenous fundamental or non-fundamental disturbances. In this paper, the financial constraint itself is the source of aggregate fluctuations. The paper establishes the existence of endogenous deterministic debt cycles in a canonical open economy model with a flow collateral constraint. Three features of the model make endogenous cycles possible: impatient households, a collateral constraint, and a pecuniary externality created by the dependence of the value of collateral on a relative price. For plausible parameter configurations, the model has a unique equilibrium exhibiting endogenous cycles in which periods of debt growth are followed by periods of debt deleveraging. In particular, three-period cycles are shown to exist, which implies by the Li-Yorke Theorem the presence of cycles of any periodicity and chaos.

Intuitively, when external debt is relatively low, the collateral constraint is slack, and impatient households embark on elevated consumption fueled by capital inflows. During this phase of the cycle, the trade balance deteriorates and the real exchange rate appreciates. In turn, the real exchange rate appreciation, by raising the value of the nontraded component of collateral, expands borrowing capacity. Eventually, debt exceeds the level that is sustainable in the long run, the collateral constraint binds, and a period of credit contraction ensues. Consumption falls, the current account reverses sign, and the real exchange rate depreciates, which exacerbates the contraction by lowering the value of the nontraded component of collateral. Once debt deleveraging has run its course, individual agents find themselves in a better financial condition and foreign lenders observing an improvement in fundamentals resume capital inflows. At this point the debt cycle starts all over again.

The endogenous credit cycles just described are inefficient because agents fail to fully internalize the welfare cost of expanding credit beyond its long-run sustainable level. In particular, agents do not take into account that their own decline in absorption has a negative effect on the value of collateral via a fall in market prices. A benevolent social planner who internalizes the pecuniary externality resolves the tradeoff between less debt expansion during the boom and larger debt deleveraging during the contraction in favor of the former. Under certain conditions, it is optimal for the planner to eliminate endogenous cycles altogether. This result provides a rationale for capital control policy even in the absence of uncertainty.

The analysis in this paper emphasizes the role of financial frictions and household demand for assets in generating endogenous instability. An interesting line of future research is to extend the analysis by introducing production, endogenous labor supply, and capital accumulation. In such an environment financial constraints would naturally impact not only intertemporal consumption decisions but also the availability of working capital for financing wages and intermediate inputs and the financing of capital investment projects. A question for future research is to ascertain whether in such an environment endogenous credit cycles of the type characterized in this paper would still emerge and how they would affect the cyclical properties of output, employment, and investment.

Appendix

This appendix contains the proofs of Propositions 3, 4, 7, and 9.

Appendix A. Proof of Proposition 3

We construct the proof through a series of lemmas and propositions. When convenient, we use the notation d for the current stock of debt, d' for next period debt, and d'' for debt in the period after the next.

Using the result that at the steady state the collateral constraint is binding (Proposition 2), we have that the steady-state level of debt, $\tilde{d}$, is given by

$$\tilde{d} = G(\tilde{d}) = \frac{\kappa(1+r)}{a(1+r)+(1-a)\kappa r} y^T .$$

Lemma A.1. *If $d < (>) \tilde{d}$ and the collateral constraint binds, then $d' > (<) d$.*

Proof. When the collateral constraint binds, we have that $d' = G(d)$. The result then follows immediately from the fact that $G' < 0$ and that, by definition, $\tilde{d} = G(\tilde{d})$. $\square$

Lemma A.2. *If $d_t < \tilde{d}$ and the collateral constraint is slack, then $d_{t+1} > d_t$.*

Proof. Suppose, on the contrary, that $d_{t+1} \le d_t$. The fact that the collateral constraint must bind in finite time (Proposition 1) and that $G(x) > x$ for any $x < \tilde{d}$ (Lemma A.1) means that debt must increase at some finite time. Thus, at some finite time when the collateral constraint is slack, the path of debt must be of the form $d \ge d' < d''$. This implies that $\Lambda(d', d) > \beta(1+r)\Lambda(d'', d')$ (recall that $\Lambda_1 < 0$ and $\Lambda_2 > 0$), which violates the Euler equation (19), since $\mu = 0$ when the collateral constraint is slack. $\square$

Proposition A.1. *The collateral constraint binds for any $d_t > \tilde{d}$. Thus, the policy function is $d_{t+1} = G(d_t)$ for any $d_t \geq \tilde{d}$.*

Proof. Suppose, on the contrary, that the collateral constraint is slack. Recall that for any $d > \tilde{d}$, $G(d) < \tilde{d}$. Thus if the collateral constraint is slack, then $d' < G(d) < \tilde{d} < d$. Then by Lemma A.1 and Lemma A.2, $d'' > d'$. By the Euler equation (19) it must be that $\mu > 0$, which is a contradiction. □

Proposition A.2. *There exist scalars d^b, $d^{b'}$, and $d^{b''}$ satisfying $d^b < \tilde{d} < d^{b'}$, $d^{b''} < \tilde{d}$, and*

$$\Lambda(d^{b'}, d^b) = \beta(1+r)\Lambda(d^{b''}, d^{b'})$$
$$d^{b'} = G(d^b)$$

and

$$d^{b''} = G(d^{b'}).$$

Proof. Let

$$H(x) \equiv \Lambda(G(x), x) - \beta(1+r)\Lambda(G(G(x)), G(x)).$$

Since $\tilde{d} = G(\tilde{d}) = G(G(\tilde{d}))$ and $\beta(1+r) < 1$, we have that

$$H(\tilde{d}) > 0.$$

Since $\Lambda_1 < 0$, $\Lambda_2 > 0$, and $G' < 0$, we have that

$$H'(x) > 0.$$

Let $\underline{x}$ be the value of x at which $y^T + G(G(x))/(1+r) - G(x) = 0$ (so that $c^{T'}$ is zero). Clearly, $\underline{x} < \tilde{d}$ and $G(\underline{x}) > \underline{x} > 0$. We then have that

$$\lim_{x \to \underline{x}} H(x) = -\infty.$$

By continuity, the above three expressions imply that there exists a value of $x < \tilde{d}$, such that $H(x) = 0$. □

Proposition A.3. *The collateral constraint binds for any $d_t \in (d^b, \tilde{d})$. Thus, the policy function is $d_{t+1} = G(d_t)$ for any $d \in (d^b, \tilde{d})$.*

Proof. Suppose, contrary to the claim, that $d_t \in (d^b, \tilde{d})$ and that the collateral constraint is slack. By Lemma A.2, we have that as long as the collateral constraint is slack, debt will grow over time. Also, by Proposition 1, the collateral constraint must bind in finite time. Let d be the level of debt in the period prior to the one in which the collateral constraint binds for the first time. Thus, we have that $d'' = G(d')$. Also, $d > d^b$, $d' < G(d) < G(d^b)$, and $d'' = G(d') > G(d^{b'}) = G(G(d^b))$. Thus, we have that

$$\begin{aligned}\Lambda(d', d) - \beta(1+r)\Lambda(d'', d') &> \Lambda(G(d^b), d^b) - \beta(1+r)\Lambda(G(G(d^b)), G(d^b)) \\ &= 0,\end{aligned}$$

which contradicts the assumption that the collateral constraint is slack. In the above expression, the inequality follows from the fact that $\Lambda_1 < 0$ and $\Lambda_2 > 0$, and the equality from the definition of d^b given in Proposition A.2. □

Proposition A.4. *The collateral constraint does not bind for any $d_t < d^b$.*

Proof. Suppose, contrary to the claim, that $d < d^b$ and that the collateral constraint binds. Since $G' < 0$, we have that $d' = G(d) > G(d^b) > d^b$. The fact that $d' > d^b$ implies, by Propositions A.1 and A.3 that the collateral constraint binds in the next period, so that $d'' = G(d') < G(d^{b\prime}) = G(G(d^b))$. We can then write

$$\begin{aligned}\Lambda(d', d) - \beta(1+r)\Lambda(d'', d') &< \Lambda(G(d^b), d^b) - \beta(1+r)\Lambda(G(G(d^b)), G(d^b)) \\ &= 0,\end{aligned}$$

which implies that the multiplier μ must be negative. In the above expression, the inequality follows from the fact that $\Lambda_1 < 0$ and $\Lambda_2 > 0$, and the equality from the definition of d^b given in Proposition A.2. □

Proposition A.5 *(Continuity and slope of the policy function). The debt policy function, $d_{t+1} = D(d_t)$, is continuous, strictly increasing for $d_t < d^b$, and strictly decreasing for $d_t > d^b$.*

Proof. Suppose that $d_t > d^b$. Then, by Propositions A.1 and A.3 we have that $D(d_t) = G(d_t)$, which is continuous and strictly decreasing under the maintained assumption that $F_1 < 1$ when the collateral constraint binds.

Suppose now that $d_t = \hat{d}_t < d^b$. By Proposition 1, the collateral constraint must bind at some finite horizon. Let the first period in which it binds be $t + \hat{J}$, where $\hat{J}$ depends on $\hat{d}_t$ in a way to be explained shortly. Then, it must be the case that $d_{t+\hat{J}+1} = G(d_{t+\hat{J}})$. For all $0 \le j \le \hat{J} - 1$, the Euler equation (19) holds with $\mu_{t+j} = 0$. Therefore, the policy function is implicitly given by the solution to

$$\begin{aligned}\Lambda(d_{t+1}, \hat{d}_t) - \beta(1+r)\Lambda(d_{t+2}, d_{t+1}) &= 0 \\ &\vdots \\ \Lambda(d_{t+\hat{J}}, d_{t+\hat{J}-1}) - \beta(1+r)\Lambda(G(d_{t+\hat{J}}), d_{t+\hat{J}}) &= 0.\end{aligned} \tag{A.1}$$

This is a system of $\hat{J}$ equations in $\hat{J}$ unknowns, $d_{t+1}, \ldots, d_{t+\hat{J}}$. Let the solution be denoted $\hat{d}_{t+1}, \ldots, \hat{d}_{t+\hat{J}}$. The policy function associated with $\hat{d}_t$ is $\hat{d}_{t+1}$, that is, $D(\hat{d}_t) = \hat{d}_{t+1}$.

Holding $\hat{J}$ fixed, the solution is continuous and differentiable at $\hat{d}_t$ because the system (A.1) is composed of the continuous and differentiable functions $\Lambda(\cdot, \cdot)$ and $G(\cdot)$. Moreover, the solution is strictly increasing at $\hat{d}_t$. To establish this property, consider a small increase in $d_{t+\hat{J}}$. The last equation of system (A.1) and the facts that $\Lambda_1 < 0$, $\Lambda_2 > 0$ and $G' < 0$ imply that $d_{t+\hat{J}-1} > \hat{d}_{t+\hat{J}-1}$ and that $\Lambda(d_{t+\hat{J}}, d_{t+\hat{J}-1})$ is larger than $\Lambda(\hat{d}_{t+\hat{J}}, \hat{d}_{t+\hat{J}-1})$. In turn, this result implies, from the penultimate equation of system (A.1) that $d_{t+\hat{J}-2} > \hat{d}_{t+\hat{J}-2}$ and that $\Lambda(d_{t+\hat{J}-1}, d_{t+\hat{J}-2}) > \Lambda(\hat{d}_{t+\hat{J}-1}, \hat{d}_{t+\hat{J}-2})$. By backward induction, it follows that $d_{t+1} > \hat{d}_{t+1}$ and that $d_t > \hat{d}_t$. We have therefore established that holding $\hat{J}$ constant, $d_{t+1}, \ldots, d_{t+\hat{J}}$ are all continuous, differentiable, and strictly increasing functions of d_t for any $d_t \le d^b$.

The first period in which the collateral constraint binds, $t + \hat{J}$, is determined by the requirement that $\hat{d}_{t+\hat{J}-1} \le d^b$ (to ensure that the economy is unconstrained in $t + \hat{J} - 1$) and that $\hat{d}_{t+\hat{J}} \ge d^b$ (to ensure that the economy is constrained in $t + \hat{J}$).

Suppose that $\hat{d}_t$ is such that $\hat{d}_{t+\hat{J}-1} < d^b$ and $\hat{d}_{t+\hat{J}} > d^b$. In this case, by continuity, $\hat{J}$ does not change for d_t in the vicinity of $\hat{d}_t$. This establishes that in this case the policy function $D(d_t)$ is continuous, differentiable, and strictly increasing at $\hat{d}_t$.

Consider now a debt level $\hat{d}_t < d^b$ such that $\hat{d}_{t+\hat{J}} = d^b$. This is a special case in which although the collateral constraint holds with equality in period $t + \hat{J}$, it does not constrain the household's choices ($\hat{\mu}_{t+\hat{J}} = 0$). In this situation, a small decline in $\hat{d}_t$ results in a change in the period in which the collateral constraint holds with equality for the first time. However, as we will see, in this case a change in the period in which the collateral constraint holds with equality for the first time does not create a discontinuity in the policy function. (Though it might create a discontinuity in its derivative.) Since by definition the economy is unconstrained in $t + \hat{J} - 1$, we have that $\hat{d}_{t+\hat{J}-1} < d^b$. Consider first a small perturbation $d_t > \hat{d}_t$. Then, if the perturbation is sufficiently small, $d_{t+\hat{J}-1} < d^b$ and $d_{t+\hat{J}} > d^b$. Therefore, $\hat{J}$ is unchanged and the policy function is right-continuous. To establish left continuity, consider a small perturbation $d_t < \hat{d}_t$. Then, $d_{t+\hat{J}} < d^b$, which means, by Proposition A.4, that the collateral constraint does not hold with equality in period $t + \hat{J}$. Therefore, the first period in which the collateral constraint holds with equality for the perturbed value of d_t must be greater than $t + \hat{J}$. We next show that this period is $t + \hat{J} + 1$. Let's examine the Euler equation in period $t + \hat{J}$ evaluated at the unperturbed allocation, that is, the one associated with $\hat{d}_t$,

$$
\begin{aligned}
0 &= \Lambda(\hat{d}_{t+\hat{J}+1}, \hat{d}_{t+\hat{J}}) - \frac{\beta(1+r)}{1-(1+r)\hat{\mu}_{t+\hat{J}}} \Lambda(G(\hat{d}_{t+\hat{J}+1}), \hat{d}_{t+\hat{J}+1}) \\
&= \Lambda(d^{b'}, d^b) - \beta(1+r)\Lambda(G(d^{b'}), d^{b'}).
\end{aligned}
$$

The second equality follows from the fact that $\hat{d}_{t+\hat{J}} = d^b$ and the definition of d^b in Proposition A.2. A small decrease in d_t results in a decline in both $d_{t+\hat{J}}$ and $d_{t+\hat{J}+1}$, so that $d_{t+\hat{J}} < d^b$ and $d_{t+\hat{J}+1} > d^b$, ensuring that period $t + \hat{J} + 1$ this is the first period in which the collateral constraint holds with equality under the perturbed value of d_t. The policy function for the perturbed allocation is given by the set of continuous functions given by the Euler equations for periods t to $t + \hat{J}$ all with $\mu_{t+j} = 0$ for $j = 0$ to $\hat{J}$. We have therefore established that the policy function is both right- and left-continuous at this particular value of $\hat{d}_t$.

Finally, consider the debt level $\hat{d}_t = d^b$. Then the policy function is $\hat{d}_{t+1} = G(\hat{d}_t)$. Let d_t be a small perturbation larger than d^b. Then the policy function is $d_{t+1} = G(d_t)$. Since $G(\cdot)$ is a continuous function, right continuity obtains. For the particular value of debt we are considering, the policy function is also implicitly given by $\Lambda(\hat{d}_{t+1}, \hat{d}_t) - \beta(1+r)\Lambda(G(\hat{d}_{t+1}), \hat{d}_{t+1}) = 0$. Let d_t be a sufficiently small perturbation less than d^b. Then we conjecture that the policy function is the solution for d_{t+1} of $\Lambda(d_{t+1}, d_t) - \beta(1+r)\Lambda(G(d_{t+1}), d_{t+1}) = 0$. To ensure that this conjecture is correct, the solution must satisfy $d_{t+1} \geq d^b$. But this is guaranteed by the continuity of $\Lambda(\cdot,\cdot)$ and $G(\cdot)$ and the fact that in the above equation $\lim_{d_t \to d^b} d_{t+1} = d^{b'} > d^b$. □

Appendix B. Proof of Proposition 4

B.1. *Characterization of the function* $G(\cdot)$

Definition B.1. Let d^U and $d^{U'}$, respectively, be the debt levels d and d' that satisfy

$$d' = F(d', d)$$

and

$$F_1(d', d) = 0.$$

From (16), we have that

$$d^U = y^T + \kappa \frac{y^T}{1+r}$$

and

$$d^{U'} = \kappa y^T.$$

Recalling that the slope of the RHS of the CC vanishes when c^T is zero, we have that d^U and $d^{U'}$ are the levels of current- and next-period debt at which the CC binds and $c^T = 0$. In this case, the only economically sensible of the two values of d' at which the CC binds is the larger one, at which the slope of the RHS of the CC is larger than 1.

Lemma B.1. $d^{U'} < \tilde{d}$.

Proof. By Proposition 2 a steady state exists and features a binding collateral constraint. That is, $F(\tilde{d}, \tilde{d}) = \tilde{d}$ and $\tilde{c}^T > 0$, where $\tilde{c}^T \equiv y^T - \frac{r\tilde{d}}{1+r}$ denotes the steady-state level of consumption. Then $\tilde{d} = F(\tilde{d}, \tilde{d}) = \kappa y^T + \kappa(1-a)/a y^{N^{1-1/\xi}} \left(\tilde{c}^T\right)^{1/\xi} > \kappa y^T = d^{U'}$. □

Lemma B.2. $d^U > \tilde{d}$.

Proof. By Proposition 2, $F(\tilde{d}, \tilde{d}) = \tilde{d}$. By Assumption 3, $F_1(\tilde{d}, \tilde{d}) < 1$. Then by convexity of $F(\cdot, \cdot)$ in its first argument, we have that $F(x, \tilde{d}) > x$ for any $x < \tilde{d}$. Since, by Lemma B.1 $d^{U'} < \tilde{d}$, we have that $F(d^{U'}, \tilde{d}) > d^{U'} = F(d^{U'}, d^U)$, where the equality follows from Definition B.1. Finally, because $F_2 < 0$, it must be that $\tilde{d} < d^U$. □

Proposition B.1. *In any equilibrium, $d < d^U$.*

Proof. The proof proceeds in four lemmas.

Lemma B.3. *If $d \geq d^U$, then $d' \neq d$.*

Proof. Suppose to the contrary that $d \geq d^U$ and that $d' = d$. Because $d^U > \tilde{d}$ (Lemma B.2), and because $F(x, x)$ is decreasing in x, we have that $d' = d \geq d^U > \tilde{d} = F(\tilde{d}, \tilde{d}) > F(d^U, d^U) \geq F(d, d) = F(d', d)$, so that the collateral constraint is violated. □

Lemma B.4. *If $d = d^U$, then $d' \not< d^U$.*

Proof. Let $d' = \kappa y^T < d^U$. The inequality follows from Lemmas B.1 and B.2. By Definition B.1, $F(\kappa y^T, d^U) = \kappa y^T$, that is, the collateral constraint is satisfied. However, at $d = d^U$ and $d' = \kappa y^T$, we have, from Definition B.1, that $c^T = 0$. Thus $d' = \kappa y^T$ cannot be an equilibrium. Similarly, because $c^T = y^T + d'/(1+r) - d^U$, $d' < \kappa y^T$ implies $c^T < 0$ and thus $d' < \kappa y^T$ cannot be an equilibrium either. Finally, show that no $d' \in (\kappa y^T, d^U)$ can be an equilibrium. For

$d' = \kappa y^T$ we have $d' = F(d', d^U)$ and for $d' = d^U$ we have $d' > F(d', d^U)$ (Lemma B.3). Because $F(d', d^U)$ is an increasing and convex function of d', it follows that $d' > F(d', d^U)$ for all $d' \in (\kappa y^T, d^U)$. □

Lemma B.5. *If $d > d^U$, then $d' \not< d$.*

Proof. Suppose first that $d' \leq \kappa y^T$, then $c^T = y^T + d'/(1+r) - d \leq y^T + \kappa y^T/(1+r) - d = d^U - d < 0$, where the equality follows from Definition B.1. But negative consumption is impossible. Now suppose that $d' \in (\kappa y^T, d^U]$. Then $d' > F(d', d^U) > F(d', d)$, so the collateral constraint is violated. The first inequality follows from Lemmas B.3 and B.4 and the second from the fact that $F_2 < 0$. Finally suppose $d' \in (d^U, d]$, then $d' > d^U > F(d^U, d^U) > F(d, d) \geq F(d', d)$, so the collateral constraint is violated. The second inequality follows from Lemma B.3, the third from the fact that $F(x, x)$ is a decreasing function of x and $d > d^U$, and the last one from the fact that $F_1 > 0$ and $d \geq d'$. □

Lemma B.6. *If $d > d^U$, then $d' \not> d$.*

Proof. Suppose on the contrary that if $d > d^U$, then $d' > d$. Then either debt will exceed the natural debt limit in finite time, which is impossible, or debt will converge to a value $\hat{d} > d^U$. In the latter case, in the limit $d' = d = \hat{d}$. But $d' = \hat{d} > d^U > F(d^U, d^U) > F(\hat{d}, \hat{d}) = F(d', d)$, so that in the limit the collateral constraint is violated. The second inequality follows from Lemma B.3 and the third inequality from the facts that $F(x, x)$ is decreasing in x and that $\hat{d} > d^U$. □

This completes the proof of Proposition B.1. □

Definition B.2. Let d^τ and $d^{\tau\prime}$ be defined as the solution for d and d', respectively, of the system

$$d' = F(d', d)$$

and

$$F_1(d', d) = 1.$$

Lemma B.7. $d^\tau < \tilde{d} < d^{\tau'}$.

Proof. We first establish that $d^\tau < \tilde{d}$. Since $F(d', d^\tau)$ is increasing and convex in d', and since by definition $F_1(d^{\tau'}, d^\tau) = 1$, and $F(d^{\tau'}, d^\tau) = d^{\tau'}$, we have that $F(x, d^\tau) > x$ for all $x \neq d^{\tau'}$. Therefore $F(\tilde{d}, d^\tau) > \tilde{d}$ unless $\tilde{d} = d^{\tau'}$. But because $F(d^{\tau'}, d^\tau) = d^{\tau'}$, $\tilde{d} = d^{\tau'}$ would require that $d^\tau = \tilde{d}$. But this cannot be the case because $F_1(\tilde{d}, \tilde{d}) < 1$, whereas $F_1(d^{\tau'}, d^\tau) = 1$. Therefore it must be that $F(\tilde{d}, d^\tau) > \tilde{d} = F(\tilde{d}, \tilde{d})$. Because $F_2 < 0$, it follows, that $d^\tau < \tilde{d}$. We now show that $d^{\tau'} > \tilde{d}$. Because $F_2 < 0$, $F(x, d^\tau) > F(x, \tilde{d}) \geq x$ for any $x \leq \tilde{d}$, where the last inequality follows from the facts that $F(x, \tilde{d})$ is increasing and convex in x, $F(\tilde{d}, \tilde{d}) = \tilde{d}$, and $F_1(\tilde{d}, \tilde{d}) < 1$. It then follows that the x such that $F(x, d^\tau) = x$ must satisfy $x > \tilde{d}$. □

Definition B.3. The function $d' = G(d)$ is defined as the solution to

$$d' = F(d', d),$$
$$F_1(d', d) < 1,$$

and

$$c^T = y^T + \frac{d'}{1+r} - d > 0.$$

The function $G(\cdot)$ is not defined for $d \le d^\tau$ or $d \ge d^U$. To see this, recall that: (a) if $d < d^\tau$, then the condition $d' = F(d', d)$ is violated for all d'; (b) if $d = d^\tau$, then the condition $d' = F(d', d)$ implies that $F_1(d', d) = 1$, which violates the condition $F_1(d', d) < 1$; and (c) if $d \ge d^U$, then the condition $d' = F(d', d)$ implies that $c^T \le 0$. For $d \in (d^\tau, d^U)$, $d' = G(d)$ is the smaller of the two solutions for d' of the equation $d' = F(d', d)$. Furthermore, $G'(\cdot) = -(1+r)F_1(G(d), d)/(1 - F_1(G(d), d)) < 0$. And finally, $\tilde{d} = G(\tilde{d})$, which follows from the fact that by definition $\tilde{d} = F(\tilde{d}, \tilde{d})$, from the assumption that $F_1(\tilde{d}, \tilde{d}) < 1$, and $\tilde{c}^T > 0$. We therefore have the following lemma:

Proposition B.2. *For $d \in (d^\tau, d^U)$, $G(d)$ is a continuous and decreasing function and satisfies $\tilde{d} = G(\tilde{d})$. For $d \le d^\tau$ or $d \ge d^U$, the function $G(\cdot)$ is not defined. The scalars d^U and d^τ are introduced in Definitions B.1 and B.2.*

This completes the characterization of the function $G(\cdot)$.

B.2. Characterization of the function $\boldsymbol{G(G(\cdot))}$

The following three lemmas give the smallest value of d for which $G(G(d))$ is well defined (that is, the collateral constraint can bind in two consecutive periods), when d is below its steady-state level $\tilde{d}$.

Lemma B.8. *If $d^{\tau'} < d^U$, then $G(G(d))$ is well defined for all $d \in (d^\tau, \tilde{d})$.*

Proof. Suppose $d^\tau < d < \tilde{d}$, then by Proposition B.2, $d' = G(d)$ exists and $d' > \tilde{d}$. Because $d > d^\tau$ and because $G'(\cdot) < 0$, it must be that $G(d) < G(d^\tau) = d^{\tau'} < d^U$. We have therefore shown that $d^\tau < G(d) < d^U$, so that $G(d')$ exists by Proposition B.2. □

Now consider the case $d^{\tau'} > d^U$.

Definition B.4. Let d^ℓ be the level of current debt d satisfying $d^U = F(d^U, d)$, where d^U is introduced in Definition B.1. From (16), we have that

$$d^\ell = y^T + \frac{d^U}{1+r} - \left[\frac{d^U - \kappa y^T}{\kappa {y^N}^{1-1/\xi}} \frac{a}{1-a}\right]^\xi.$$

Lemma B.9. *If $d^{\tau'} > d^U$, then $d^\tau < d^\ell < \tilde{d}$.*

Proof. We first establish that $d^\tau < d^\ell$. Because $F(\cdot, \cdot)$ is increasing and convex in its first argument, we have from Definition B.2 that $F(x, d^\tau) > x$ for all $x < d^{\tau'}$. Because $d^U < d^{\tau'}$, we have that $F(d^U, d^\tau) > d^U = F(d^U, d^\ell)$, where the equality follows from Definition B.4. Because $F_2 < 0$, it follows that $d^\tau < d^\ell$. We now establish that $\tilde{d} > d^\ell$. Consider $x \in (\tilde{d}, d^{\tau'})$. From Lemma B.7, this interval is non-empty. Then for any $x \in (\tilde{d}, d^{\tau'})$, $F(x, \tilde{d}) < x < F(x, d^\tau)$. The first equality follows from the assumption that $F_1(\tilde{d},\tilde{d}) < 1$ and the second from the fact that $F_1(d^{\tau\prime}, d^{\tau}) = 1$. Because $\tilde{d} < d^U < d^{\tau'}$, $F(d^U,\tilde{d}) < d^U = F(d^U, d^{\ell})$. Finally, since $F_2 < 0$, we have that $\tilde{d} > d^{\ell}$. □

Lemma B.10. *If $d^{\tau'} > d^U$ and $d^{\ell} < d < \tilde{d}$, then $G(G(d))$ exists.*

Proof. Suppose $d^{\ell} < d < \tilde{d}$, then, by Lemma B.9, $d \in (d^{\tau}, \tilde{d})$. By Proposition B.2, $d' = G(d)$ exists and $d' > \tilde{d}$ and because $d > d^{\ell}$, $G(d) < G(d^{\ell}) = d^U$. We have therefore shown that $d^{\tau} < G(d) < d^U$, so that $G(G(d))$ exists by Proposition B.2. □

This completes the characterization of the function $G(G(\cdot))$.

B.3. Existence of the debt threshold $\boldsymbol{d^b}$

Assumption B.1. If $d^{\tau\prime} < d^U$, then

$$\lim_{x\searrow d^{\tau}} [\Lambda(G(x),x) - \beta(1+r)\Lambda(G(G(x)),G(x))] < 0.$$

Proposition B.3. *There exist scalars d^b, $d^{b'}$, and $d^{b''}$ satisfying $d^b < \tilde{d} < d^{b'}$, $d^{b''} < \tilde{d}$, and*

$$\Lambda(d^{b'}, d^b) = \beta(1+r)\Lambda(d^{b''}, d^{b'})$$
$$d^{b'} = G(d^b)$$

and

$$d^{b''} = G(d^{b'}).$$

Proof. Let

$$H(x) \equiv \Lambda(G(x),x) - \beta(1+r)\Lambda(G(G(x)),G(x)).$$

Since $\tilde{d} = G(\tilde{d}) = G(G(\tilde{d}))$ and $\beta(1+r) < 1$, we have that

$$H(\tilde{d}) > 0.$$

Since $\Lambda_1 < 0$, $\Lambda_2 > 0$, and $G' < 0$, we have that

$$H'(x) > 0.$$

Suppose first that $d^{\tau\prime} > d^U$. Then, recalling that $y^T + G(G(d^{\ell}))/(1+r) - G(d^{\ell}) = 0$ (so that $c^{T'} = 0$) and that $d^{\ell} < \tilde{d}$, we have that

$$\lim_{x\searrow d^{\ell}} H(x) = -\infty.$$

Since $d^{\ell} < \tilde{d}$, by continuity, the above three expressions imply that there exists a value of $x < \tilde{d}$, such that $H(x) = 0$.

Suppose now that $d^{\tau\prime} < d^U$. Then, by Assumption B.1, we have that

$$\lim_{x\searrow d^{\tau}} H(x) < 0.$$

Since $d^{\tau} < \tilde{d}$, we have that in this case too there exists a value of $x < \tilde{d}$, such that $H(x) = 0$. □

The proof of Proposition 4 is then identical to that of Proposition 3 (Cobb-Douglas aggregator), with Proposition B.3 taking the place of Proposition A.2.

Appendix C. Proof of Proposition 7

We begin by reproducing the statement of the Li and Yorke (1975) theorem commonly known as 'period three implies chaos:'

Theorem 1 *(Li and Yorke (1975)). Let J be an interval and let $D : J \to J$ be continuous. Assume there is a point $d \in J$ for which the points $d' = D(d)$, $d'' = D^2(d)$, and $d''' = D^3(d)$, satisfy*

$$d''' \leq d < d' < d''.$$

Then, for every $k = 1, 2, \ldots$ there is a periodic point in J having period k. Furthermore, there is an uncountable set $S \subset J$ (containing no periodic points), which satisfies the following conditions:

1. *For every $p, q \in S$ with $p \neq q$, $\limsup_{n\to\infty} |D^n(p) - D^n(q)| > 0$ and $\liminf_{n\to\infty} |D^n(p) - D^n(q)| = 0$.*
2. *For every $p \in S$ and periodic point $q \in J$, $\limsup_{n\to\infty} |D^n(p) - D^n(q)| > 0$.*

Li and Yorke remark that if there is a periodic point with period 3, then the hypothesis $d''' \leq d < d' < d''$ is satisfied. Section 5.4 shows the existence of plausible calibrations for which the model economy has a three-period debt cycle. Further, Proposition 3 establishes that the policy function D is continuous. It remains to show that there is an interval J such that $D : J \to J$. The following lemma establishes this result.

Lemma C.1. *The policy function $d' = D(d)$ maps the interval $[d^{b''}, d^{b'}]$ into itself.*

Proof. Proposition 3 shows that $D(d)$ is continuous, increasing for $d < d^b$, and decreasing for $d > d^b$. Thus, $D(d) \leq D(d^b) = d^{b'}$. Suppose now that $d \in [d^{b''}, d^b)$. By Proposition 3, in this range $D(d) > d$, which implies that $D(d) > d^{b''}$. Finally, if $d \in (d^b, d^{b'}]$, we have, by the same proposition, that $D(d) = G(d)$. Since $G(\cdot)$ is decreasing, we have that $D(d) = G(d) \geq G(d^{b'}) = d^{b''}$. □

Appendix D. Proof of Proposition 9

The proof proceeds in three lemmas.

Lemma D.1. *Suppose $\beta(1+r) < 1$, $\xi = 1$, $F_1(\tilde{d}, \tilde{d}) = \frac{\kappa(1-a)}{a(1+r)} < 1$, and $F_1(\tilde{d}, \tilde{d}) > 1/[1+\beta(1+r)]$. Then the debt policy function in the Ramsey equilibrium, $D^r(\cdot)$, satisfies $D^r(d_t) = \tilde{d}$ for any $d_t \in [d^{b1}, \tilde{d})$, with d^{b1} implicitly given by $\Lambda(\tilde{d}, d^{b1}) = \beta(1+r)\Lambda(\tilde{d}, \tilde{d})$.*

Proof. We wish to characterize a debt threshold $d^{b1} < \tilde{d}$ with the property that if $d_t \in [d^{b1}, \tilde{d})$, then the Ramsey economy reaches the steady state $\tilde{d}$ in one period, that is, $D^r(d_t) = \tilde{d}$ for all $d_t \in [d^{b1}, \tilde{d})$. For this conjecture to be correct, the complete set of Ramsey equilibrium conditions, equations (33)-(38), must be satisfied for all periods greater than or equal to t. Suppose that $d_t \in [d^{b1}, \tilde{d})$. Then, by the conjecture, $d_{t+j} = \tilde{d}$ for all $j > 0$.

Consider first equation (37). We have already shown that in the steady state consumption is positive. Thus, we only need to ascertain whether consumption in period t is positive. Note that $c_t^T = y^T + \tilde{d}/(1+r) - d_t > y^T + \tilde{d}/(1+r) - \tilde{d} > 0$. Consider next the collateral constraint (34). We already established that in the steady state of the Ramsey economy the collateral constraint holds with equality (Proposition 8). So we only need to check that it is satisfied in period t. Note that $d_{t+1} = \tilde{d} = G(\tilde{d}) < G(d_t)$. The first equality follows from the conjecture and the inequality from the fact that $G'(\cdot) < 0$ and $d_t < \tilde{d}$. Consider now the transversality condition (38). It is satisfied under the conjectured path because d_{t+j} is constant for all $j > 0$. Consider next the complementary slackness condition (36). Because the collateral constraint is slack in period t, $\mu_t^R = 0$, so the slackness condition is satisfied in period t. In period $t + j$ for $j > 0$, the collateral constraint holds with equality, $\tilde{d} = F(\tilde{d}, \tilde{d})$, therefore (36) also holds. It remains to show that under the conjecture the Euler equations (33) and the non-negativity condition (35) are satisfied. We have already shown that $\mu_t^R = 0$. Then the Euler equation in period t takes the form

$$\Lambda(\tilde{d}, d_t) = \beta(1+r)\frac{\Lambda(\tilde{d}, \tilde{d})}{1 - (1+r)F_1\mu_{t+1}^R}. \tag{D.1}$$

We omit the arguments of F_1 because when $\xi = 1$, F_1 is a constant. Because $\Lambda_2 > 0$, μ_{t+1}^R is strictly increasing in d_t. Let d^{b1} be the smallest level of debt for which μ_{t+1}^R is non-negative. We then have that d^{b1} is implicitly given by $\Lambda(\tilde{d}, d^{b1}) = \beta(1+r)\Lambda(\tilde{d}, \tilde{d})$. Clearly, $d^{b1} < \tilde{d}$, and $\mu_{t+1}^R = 0$ when $d_t = d^{b1}$. As $d_t \to \tilde{d}$, the Euler equation (D.1) implies that μ_{t+1}^R satisfies $\mu_{t+1}^R \to \frac{1-\beta(1+r)}{(1+r)F_1} > 0$. In period $t + j$, for $j \geq 1$, the Euler equation (33) becomes

$$[1 - (1+r)F_1\mu_{t+j+1}^R] = \frac{\beta(1+r)}{1 - (1+r)\mu_{t+j}^R}[1 - (1+r)F_1\mu_{t+j}^R],$$

which determines μ_{t+j+1}^R as a function of μ_{t+j}^R. It is convenient to introduce the variable transformation $s_{t+j} \equiv 1 - (1+r)F_1\mu_{t+j}^R$. Then, the Euler equation in period $t + j$, for $j \geq 1$, can be written as

$$s_{t+j+1} = \frac{\beta(1+r)F_1 s_{t+j}}{F_1 - 1 + s_{t+j}} \equiv \pi(s_{t+j}), \tag{D.2}$$

with $\pi'(\cdot) < 0$. Note that $\pi(s_{t+j})$ has a discontinuity at $s_{t+j} = 1 - F_1$ and converges to infinity (minus infinity) as s_{t+j} approaches $1 - F_1$ from the right (left). The nonnegativity condition on μ_{t+j}^R, equation (35), restricts $s_{t+j} \leq 1$. The initial condition s_{t+1} ranges continuously from $\beta(1+r)$ when $d_t \to \tilde{d}$ to 1 when $d_t = d^{b1}$. It follows that a necessary condition for $s_{t+2} \leq 1$ for any $d_t \in [d^{b1}, \tilde{d})$ is

$$\beta(1+r) > 1 - F_1. \tag{D.3}$$

The difference equation (D.2) has two steady states, $s_{t+j} = 0$ and $s_{t+j} = 1 - F_1[1 - \beta(1+r)] > 1 - F_1$. The first steady state lies outside the range of initial conditions for s_{t+1}, $[\beta(1+r), 1]$, and the second steady state lies inside. This fact together with $\pi'(\cdot) < 0$, implies that the maximum possible value of s_{t+j+1}, for all $j \geq 1$, is

$$\max\{\pi(\beta(1+r)), \pi^2(1)\} = \pi(\beta(1+r)) = \frac{\beta^2(1+r)^2 F_1}{F_1 - 1 + \beta(1+r)}.$$

The first equality follows from the fact that $\pi(1) = \beta(1+r)$. Thus $\frac{\beta^2(1+r)^2 F_1}{F_1 - 1 + \beta(1+r)}$ must be less than unity if μ^R_{t+j+1} is to be nonnegative for all $j \geq 1$. This will be the case if

$$F_1 > \frac{1}{1+\beta(1+r)}.$$

This restriction is more stringent than the one given in (D.3). □

We have defined d^{b1} as

$$\Lambda(\tilde{d}, d^{b1}) - \gamma \Lambda(\tilde{d}, \tilde{d}) = 0.$$

We now generalize this definition.

Definition D.1. Let d^{bi}, for $i \geq 2$, be given by

$$\Lambda(d^{bi-1}, d^{bi}) - \gamma \Lambda(d^{bi-2}, d^{bi-1}) = 0,$$

with $d^{b0} \equiv \tilde{d}$.

Clearly, $d^{bi} < d^{bi-1}$ for any $i \geq 1$.

Lemma D.2. *Suppose $\beta(1+r) < 1$, $\xi = 1$, $F_1(\tilde{d}, \tilde{d}) = \frac{\kappa(1-a)}{a(1+r)} < 1$, and $F_1(\tilde{d}, \tilde{d}) > 1/[1+\beta(1+r)]$. Then, for any $d_t < d^{b1}$, there exists an integer $i \geq 1$ such that the Ramsey equilibrium path of debt is of the form $(d_t, d_{t+1}, d_{t+2}, \ldots, d_{t+i}, \tilde{d}, \tilde{d}, \ldots)$, with $d_t < d_{t+1} < d_{t+2} < \cdots < d_{t+i} < \tilde{d}$.*

Proof. Suppose that $d_t \in [d^{bi+1}, d^{bi})$ for $i \geq 1$. Conjecture that debt converges to $\tilde{d}$ in $i+1$ periods, and that its equilibrium path, denoted $(d_t, d_{t+1}, d_{t+2}, \ldots, d_{t+i}, \tilde{d}, \tilde{d}, \ldots)$, is given by the solution of

$$\Lambda(d_{t+i}, d_t) - \gamma \Lambda(\tilde{d}, d_{t+i}) = 0, \tag{D.4}$$

if $i = 1$, and

$$\begin{aligned} \Lambda(d_{t+i}, d_{t+i-1}) - \gamma \Lambda(\tilde{d}, d_{t+i}) &= 0 \\ &\vdots \\ \Lambda(d_{t+1}, d_t) - \gamma \Lambda(d_{t+2}, d_{t+1}) &= 0 \end{aligned} \tag{D.5}$$

if $i \geq 2$. Clearly, $d_{t+k} \in [d^{b(i-k+1)}, d^{b(i-k)})$, for all $k = 1, \ldots, i$, so the conjectured convergence to $\tilde{d}$ is monotonic. The collateral constraint (34) is satisfied with strict inequality along the proposed equilibrium path since $d_t < d_{t+1} < d_{t+2} < \cdots < d_{t+i} < \tilde{d}$ and $G(x) > \tilde{d}$ for any $x < \tilde{d}$. Because the collateral constraint holds with inequality, satisfaction of the slackness condition (36) requires that $\mu^R_t, \mu^R_{t+1}, \ldots, \mu^R_{t+i} = 0$. This implies that (35) holds. The systems (D.4) and (D.5) together with the fact that $\mu^R_{t+k} = 0$ for $k = 1, \ldots, i$ guarantee that the Euler equation (33) is satisfied. Along the conjectured solution, consumption is positive so that (37) is satisfied. To see this note that $c^T_{t+k} = y^T + d_{t+k+1}/(1+r) - d_{t+k} > y^T - r d_{t+k}/(1+r) > y^T - r/(1+r)\tilde{d} > 0$ for all $k = 1, \ldots, i$. Finally, debt is bounded above by $\tilde{d}$, so that the transversality condition (38) holds. This establishes that the conjectured solution is indeed the Ramsey equilibrium path. □

Lemma D.3. *Suppose $\beta(1+r) < 1$, $\xi = 1$, $F_1(\tilde{d}, \tilde{d}) = \frac{\kappa(1-a)}{a(1+r)} < 1$, and $F_1(\tilde{d}, \tilde{d}) > 1/[1+\beta(1+r)]$. Then, for any $d_t > \tilde{d}$, there exists an integer $i \geq 0$ such that the Ramsey equilibrium path of debt is of the form $(d_t, d_{t+1}, d_{t+2}, \ldots, d_{t+i}, \tilde{d}, \tilde{d}, \ldots)$, with $d_t > d_{t+1} < d_{t+2} < \cdots < d_{t+i} < \tilde{d}$.*

Proof. The proof consists in conjecturing that the collateral constraint binds in period t, so that $d_{t+1} = G(d_t)$. Since $d_t > \tilde{d} = G(\tilde{d})$ and $G'(\cdot) < 0$, we have that $d_{t+1} < \tilde{d}$. So we know from Proposition 9 that starting in $t+1$ the economy converges monotonically and in finite time to $\tilde{d}$. In particular, we have that $d_{t+2} > d_{t+1}$. So, from Lemmas D.1 and D.2 we have that the collateral constraint is slack in period $t+1$, so that $\mu_{t+1}^R = 0$. The Euler condition in period t is then given by

$$\frac{\Lambda(d_{t+1}, d_t)}{1-(1+r)F_1\mu_t^R}(1-(1+r)\mu_t^R) = \beta(1+r)\Lambda(d_{t+2}, d_{t+1}),$$

which implies that $\mu_t^R > 0$ because $d_t, d_{t+2} > d_{t+1}$ implies that $\Lambda(d_{t+1}, d_t) > \Lambda(d_{t+2}, d_{t+1})$. This completes the proof. $\square$

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