Technical Appendix to:
Heterogeneous Downward Nominal Wage Rigidity: Foundations of a Nonlinear Phillips Curve

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In this appendix we derive the equilibrium conditions of the NK model with sticky wages used in the body of the paper as a point of comparison with the HDNWR model. Note that the appendix first develops a general NK model with both price and wage stickiness. Then to obtain the NK model with flexible prices but sticky wages that we use in the body of the paper, we set the parameters ϵ_p and θ_p, to be introduced below, to ∞ and 0, respectively.

1 An NK Model with Price and Wage Rigidity

1.1 Households

Preferences of the household are the same as in the HDNWR model with endogenous labor supply, which we repeat here for convenience

$$E_0 \sum_{t=0}^{\infty} \beta^t e^{\xi_t} \left[U(c_t) - \int_0^1 V(h_{it}) di \right], \tag{1}$$

where h_{it} denotes hours of variety i actually worked by the household and ξ_t is an exogenous preference shock. The budget constraint of the household is also the same as in the HDNWR model,

$$P_t c_t + \frac{B_t}{1+i_t} = \int_0^1 W_{it} h_{it} di + B_{t-1} + \Phi_t, \tag{2}$$

where Φ_t denotes profit income from the ownership of firms. Households take the number of hours of each variety worked as given. As is standard under Calvo-type nominal wage rigidity, the nominal wage is assumed to be set by a union and the household must satisfy labor demand at the posted wage. Households do choose consumption (c_t) and nominal bond holdings (B_t). The associated first-order conditions are

$$\lambda_t = e^{\xi_t} U'(c_t)$$

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and

$$\lambda_t = \beta(1+i_t)E_t\left\{\frac{\lambda_{t+1}}{1+\pi_{t+1}}\right\}$$

where λ_t denotes the marginal utility of income.

Consumption is assumed to be an aggregate of intermediate consumption goods denoted c_{it},

$$c_t = \left(\int_0^1 c_{it}^{1-\frac{1}{\epsilon_p}}\right)^{\frac{1}{1-\frac{1}{\epsilon_p}}},$$

where $\epsilon_p > 1$ denotes the elasticity of substitution across varieties of consumption goods. The parameter ϵ_p is given the subscript p because, as we shall see shortly, it plays a role for the way nominal price rigidity is introduced into this model.

The cost minimizing demand for variety i, given c_t is

$$c_{it} = \left(\frac{P_{it}}{P_t}\right)^{-\epsilon_p} c_t, \tag{3}$$

where P_{it} denotes the nominal price of the intermediate consumption good of variety i and P_t is given by

$$P_t = \left(\int_0^1 P_{it}^{1-\epsilon_p} di\right)^{\frac{1}{1-\epsilon_p}}. \tag{4}$$

This definition guarantees that P_t is the minimum cost of one unit of the composite consumption good, that is,

$$P_t c_t = \int_0^1 P_{it} c_{it} di.$$

1.2 Firms

Intermediate goods of variety i are produced by monopolistically competitive firms with the production technology

$$y_{it} = z_t F(n_{it}), \tag{5}$$

where n_{it} denotes the amount of an aggregate labor index demanded by firm i. We will show shortly how n_{it} is related to the varieties of labor supplied by households (h_{jt}). The production function F is assumed to be increasing and concave. In particular, we assume that

$$F(n) = n^\alpha,$$

with $\alpha \in (0,1]$. Firms can hire n_{it} at the nominal wage W_t, which they take as given. Firms face the demand given in equation (3) and choose P_{it} taking P_t and c_t as given. The profits of firm i in period t are equal to

$$P_{it}y_{it} - W_t n_{it}.$$

At posted prices firms must satisfy demand, which implies that production has to satisfy

$$y_{it} = z_t F(n_{it}) \geq \left(\frac{P_{it}}{P_t}\right)^{-\epsilon_p} c_t. \tag{6}$$

Each period with probability $1-\theta_p$ the firm can set the price freely and with probability θ_p it must charge the same price as in the previous period, where $\theta_p \in (0,1)$. The parameter θ_p controls the degree of nominal price rigidity. If $\theta_p = 0$ prices are fully flexible and if $\theta_p \rightarrow 1$, then prices are fixed.

Suppose in period t firm i can reset prices. Let $\tilde{P}_{it}$ denote the price the firm chooses. The firm chooses $\tilde{P}_{it}$ to maximize the expected profits it makes while stuck with this price subject to the constraint that at posted prices it satisfies demand. The firm is assumed to discount future profits back to period t with the discount factor $\beta^k \lambda_{t+k}/\lambda_t$. Formally, the firm picks $\tilde{P}_{it}$ to maximize

$$E_t \sum_{k=0}^{\infty} (\beta\theta_p)^k \frac{\lambda_{t+k}}{\lambda_t} \left\{ \frac{\tilde{P}_{it}}{P_{t+k}} \left(\frac{\tilde{P}_{it}}{P_{t+k}} \right)^{-\epsilon_p} c_{t+k} - \frac{W_{t+k}}{P_{t+k}} n_{i,t+k} + mc_{i,t+k} \left[z_{t+k} F(n_{i,t+k}) - \left(\frac{\tilde{P}_{it}}{P_{t+k}} \right)^{-\epsilon_p} c_{t+k} \right] \right\},$$

where $mc_{i,t+k}$ denotes the Lagrange multiplier on (6). The first-order condition with respect to $n_{i,t+k}$ is static

$$mc_{i,t+k} = \frac{\frac{W_{t+k}}{P_{t+k}}}{z_{t+k} F'(n_{i,t+k})}$$

and the first-order condition with respect to $\tilde{P}_{it}$, after some rearranging, can be expressed as

$$E_t \sum_{k=0}^{\infty} (\beta\theta_p)^k \frac{\lambda_{t+k}}{\lambda_t} z_{t+k} F(n_{i,t+k}) \left[\frac{\tilde{P}_{it}}{P_{t+k}} - \frac{\epsilon_p}{\epsilon_p - 1} \frac{W_{t+k}/P_{t+k}}{z_{t+k} F'(n_{i,t+k})} \right] = 0 \tag{7}$$

Next, we want to express this first-order condition in recursive form and we wish to show that all firms that get to re-optimize the price in period t choose the same price, that is, $\tilde{P}_{it}$, is the same for all firms i that get to pick the price. Notice that under profit maximization (6) holds with equality. It then follows that n_{it+k} depends on $\tilde{P}_{it}$ and aggregate variables and therefore the same is true for mc_{it+k}. It follows that all firms that get to re-optimize the price in period t will pick the same price. So we drop the subscript i. Use (6) holding with equality to eliminate $z_{t+k}F(n_{i,t+k})$, that is, in the first-order condition (7) replace $z_{t+k}F(n_{i,t+k})$ with $\left(\frac{\tilde{P}_t}{P_{t+k}}\right)^{-\epsilon_p} c_{t+k}$.

Next introduce two auxiliary variables, $x_{1,t}$ and $x_{2,t}$ to be able to write the first-order condition (7) recursively. The first step is to write (7) as

$$x_{1,t} - x_{2,t} = 0,$$

where

$$x_{1,t} \equiv E_t \sum_{k=0}^{\infty} (\beta\theta_p)^k \frac{\lambda_{t+k}}{\lambda_t} \left(\frac{\tilde{P}_t}{P_{t+k}} \right)^{-\epsilon_p} c_{t+k} \frac{\tilde{P}_t}{P_{t+k}}$$

and

$$x_{2,t} \equiv E_t \sum_{k=0}^{\infty} (\beta\theta_p)^k \frac{\lambda_{t+k}}{\lambda_t} \left(\frac{\tilde{P}_t}{P_{t+k}} \right)^{-\epsilon_p} c_{t+k} \frac{\epsilon_p}{\epsilon_p - 1} \frac{W_{t+k}/P_{t+k}}{z_{t+k} F'(n_{i,t+k})}.$$

The second step is to express $x_{1,t}$ and $x_{2,t}$ recursively. For $x_{1,t}$ this is pretty straightforward:

$$\begin{aligned} x_{1,t} &\equiv E_t \sum_{k=0}^{\infty} (\beta\theta_p)^k \frac{\lambda_{t+k}}{\lambda_t} \left(\frac{\tilde{P}_t}{P_{t+k}} \right)^{-\epsilon_p} c_{t+k} \frac{\tilde{P}_t}{P_{t+k}} \\ &= \left(\frac{\tilde{P}_t}{P_t} \right)^{1-\epsilon_p} c_t + E_t \sum_{k=1}^{\infty} (\beta\theta_p)^k \frac{\lambda_{t+k}}{\lambda_t} \left(\frac{\tilde{P}_t}{P_{t+k}} \right)^{-\epsilon_p} c_{t+k} \frac{\tilde{P}_t}{P_{t+k}} \\ &= \left(\frac{\tilde{P}_t}{P_t} \right)^{1-\epsilon_p} c_t + \beta\theta_p E_t \frac{\lambda_{t+1}}{\lambda_t} \left(\frac{\tilde{P}_t}{\tilde{P}_{t+1}} \right)^{1-\epsilon_p} x_{1,t+1} \\ &= \tilde{p}_t^{1-\epsilon_p} c_t + \beta\theta_p E_t \frac{\lambda_{t+1}}{\lambda_t} \left(\frac{\tilde{p}_t}{(1+\pi_{t+1})\tilde{p}_{t+1}} \right)^{1-\epsilon_p} x_{1,t+1} \end{aligned}$$

where we used

$$\tilde{p}_t \equiv \frac{\tilde{P}_t}{P_t}$$

and

$$1 + \pi_{t+1} \equiv \frac{P_{t+1}}{P_t}.$$

But for $x_{2,t}$ it is less straightforward. In fact, to express $x_{2,t}$ recursively we make use of the assumption that $F(n) = n^\alpha$. This makes things easier because then we can solve condition (6) holding with equality for n_{it+k} in terms of $\tilde{P}_t$. Specifically, we have that

$$n_{it+k} = \left[\frac{\left(\frac{\tilde{P}_{it}}{P_{t+k}} \right)^{-\epsilon_p} c_{t+k}}{z_{t+k}} \right]^{\frac{1}{\alpha}}$$

and

$$F'(n_{it+k}) = \alpha \left[\frac{\left(\frac{\tilde{P}_{it}}{P_{t+k}} \right)^{-\epsilon_p} c_{t+k}}{z_{t+k}} \right]^{\frac{\alpha-1}{\alpha}}$$

Thus far, we have used the notation $n_{i,t+k}$ to indicate the number of hours by firm i that last reoptimized its price in period t. To make this clear, let's use the notation $n_{t+k|t}$ for a firm that last optimized in t and $n_{t+k|t+1}$ for a firm that last reoptimized in period $t+1$. With this notation we can write $x_{2,t+1}$ as follows

$$x_{2,t+1} = E_{t+1} \sum_{h=0}^{\infty} (\beta\theta_p)^h \frac{\lambda_{t+1+h}}{\lambda_{t+1}} \left(\frac{\tilde{P}_{t+1}}{P_{t+1+h}} \right)^{-\epsilon_p} c_{t+1+h} \frac{\epsilon_p}{\epsilon_p - 1} \frac{W_{t+1+h}/P_{t+1+h}}{z_{t+1+h} F'(n_{t+1+h|t+1})}$$

and

$$F'(n_{it+k|t}) = \left(\frac{\tilde{P}_t}{\tilde{P}_{t+1}} \right)^{-\epsilon_p + \epsilon_p/\alpha} F'(n_{it+k|t+1}).$$

Now write $x_{2,t}$ as

$$\begin{aligned} x_{2,t} &= \left(\frac{\tilde{P}_t}{P_t}\right)^{-\epsilon_p} c_t \frac{\epsilon_p}{\epsilon_p - 1} \frac{W_t/P_t}{z_t F'(n_{t|t})} + E_t \sum_{k=1}^{\infty} (\beta\theta_p)^k \frac{\lambda_{t+k}}{\lambda_t} \left(\frac{\tilde{P}_t}{P_{t+k}}\right)^{-\epsilon_p} c_{t+k} \frac{\epsilon_p}{\epsilon_p - 1} \frac{W_{t+k}/P_{t+k}}{z_{t+k} F'(n_{t+k|t})} \\ &= \left(\frac{\tilde{P}_t}{P_t}\right)^{-\epsilon_p} c_t \frac{\epsilon_p}{\epsilon_p - 1} \frac{W_t/P_t}{z_t F'(n_{t|t})} + \beta\theta_p E_t \frac{\lambda_{t+1}}{\lambda_t} \left(\frac{\tilde{P}_t}{\tilde{P}_{t+1}}\right)^{-\epsilon_p/\alpha} x_{2,t+1} \\ &= \tilde{p}_t^{-\epsilon_p} c_t \frac{\epsilon_p}{\epsilon_p - 1} \frac{W_t/P_t}{z_t F'(n_{t|t})} + \beta\theta_p E_t \frac{\lambda_{t+1}}{\lambda_t} \left(\frac{\tilde{p}_t}{(1+\pi_{t+1})\tilde{p}_{t+1}}\right)^{-\epsilon_p/\alpha} x_{2,t+1} \end{aligned}$$

Finally, let $\tilde{n}_t$ denote the number of hours hired in period t by a firm that reoptimized prices in period t, which is implicitly given by the solution to

$$z_t F(\tilde{n}_t) = \tilde{p}_t^{-\epsilon_p} c_t.$$

Let

$$w_t \equiv \frac{W_t}{P_t}$$

In sum, we can express the first-order condition of the firm, equation (7), recursively and equation (6) holding with equality with the following four equations:

$$x_{1,t} = x_{2,t} \tag{8}$$

$$x_{1,t} = \tilde{p}_t^{1-\epsilon_p} c_t + \beta\theta_p E_t \frac{\lambda_{t+1}}{\lambda_t} \left(\frac{\tilde{p}_t}{(1+\pi_{t+1})\tilde{p}_{t+1}}\right)^{1-\epsilon_p} x_{1,t+1} \tag{9}$$

$$x_{2,t} = \tilde{p}_t^{-\epsilon_p} c_t \frac{\epsilon_p}{\epsilon_p - 1} \frac{w_t}{z_t F'(\tilde{n}_t)} + \beta\theta_p E_t \frac{\lambda_{t+1}}{\lambda_t} \left(\frac{\tilde{p}_t}{(1+\pi_{t+1})\tilde{p}_{t+1}}\right)^{\frac{-\epsilon_p}{\alpha}} x_{2,t+1} \tag{10}$$

$$z_t F(\tilde{n}_t) = \tilde{p}_t^{-\epsilon_p} c_t \tag{11}$$

Note that either a firm charges the same price as last period or it charges $\tilde{P}_t$. From the definition of the price index, equation (4), we then have

$$\begin{aligned} P_t^{1-\epsilon_p} &= \int_0^1 P_{it}^{1-\epsilon_p} di \\ P_t^{1-\epsilon_p} &= \int_{\theta_p} P_{it-1}^{1-\epsilon_p} di + \int_{1-\theta_p} \tilde{P}_t^{1-\epsilon_p} di \\ P_t^{1-\epsilon_p} &= \theta_p P_{t-1}^{1-\epsilon_p} + (1-\theta_p)\tilde{P}_t^{1-\epsilon_p} \\ 1 &= \theta_p (1+\pi_t)^{\epsilon_p - 1} + (1-\theta_p)\tilde{p}_t^{1-\epsilon_p} \end{aligned}$$

1.3 Firms — alllowing for price indexation

We now assume that when firms cannot reoptimize the price they get to adjust it according to the following rule of thumb. Let $\tilde{P}_{it,t+k}$ denote the price of a firm in period $t+k$ that got to reoptimize its price for the last time in period t. We assume that

$$\tilde{P}_{it,t+k} = X_{t,t+k}\tilde{P}_{it}; \quad \text{with } X_{t,t+k} \equiv \left(\Pi_{j=0}^{k-1} (1+\pi^*)^{\chi^*} (1+\pi_{t+j})^{\chi^p}\right) \tag{12}$$

where $\tilde{P}_{it}$ is the price the firm picks in period t when it has the chance to reoptimize in period t. The parameters $\chi^* \geq 0$ and $\chi^p \geq 0$ control the degree of indexation. If $\chi^* = \chi^p = 0$, then there is no indexation, the case we studied in the previous section. If $\chi^* + \chi^p = 1$, then we say there is full indexation and the size of χ^* controls the degree of indexation to steady state inflation relative to indexation to lagged inflation. If $0 < \chi^* + \chi^p < 1$, then we have partial indexation. Notice that since the firm takes $X_{t,t+k}$ as exogenously given, the problem faced by a firm that gets to reoptimize prices in period t still consists in choosing just $\tilde{P}_{it}$.

The following relationship between $X_{t,t+1+h}$ and $X_{t+1,t+1+h}$ will be useful in what follows For any $h \geq 0$, we have

$$\begin{aligned} \frac{X_{t,t+1+h}}{X_{t+1,t+1+h}} &= \frac{\Pi_{j=0}^{h}(1+\pi^*)^{\chi^*}(1+\pi_{t+j})^{\chi^p}}{\Pi_{j=0}^{h-1}(1+\pi^*)^{\chi^*}(1+\pi_{t+1+j})^{\chi^p}} \\ &= (1+\pi^*)^{\chi^*}(1+\pi_t)^{\chi^p} \end{aligned}$$

Suppose in period t firm i can reset prices. The firm picks $\tilde{P}_{it}$ to maximize

$$E_t \sum_{k=0}^{\infty} (\beta\theta_p)^k \frac{\lambda_{t+k}}{\lambda_t} \left\{ \frac{\tilde{P}_{it,t+k}}{P_{t+k}} \left(\frac{\tilde{P}_{it,t+k}}{P_{t+k}} \right)^{-\epsilon_p} c_{t+k} - \frac{W_{t+k}}{P_{t+k}} n_{i,t+k} + mc_{i,t+k} \left[z_{t+k} F(n_{i,t+k}) - \left(\frac{\tilde{P}_{it,t+k}}{P_{t+k}} \right)^{-\epsilon_p} c_{t+k} \right] \right\},$$

where $\tilde{P}_{it,t+k}$ is given by equation (12). The first-order condition with respect to $n_{i,t+k}$ is the same as in the case without indexation

$$mc_{i,t+k} = \frac{\frac{W_{t+k}}{P_{t+k}}}{z_{t+k} F'(n_{i,t+k})}.$$

But the first-order condition with respect to $\tilde{P}_{it}$ now is slightly different, where we used to have $\tilde{P}_{it}$ we now have $X_{t,t+k}\tilde{P}_{it}$

$$E_t \sum_{k=0}^{\infty} (\beta\theta_p)^k \frac{\lambda_{t+k}}{\lambda_t} \left(\frac{X_{t,t+k}\tilde{P}_{it}}{P_{t+k}} \right)^{-\epsilon_p} c_{t+k} \left[\frac{X_{t,t+k}\tilde{P}_{it}}{P_{t+k}} - \frac{\epsilon_p}{\epsilon_p - 1} \frac{W_{t+k}/P_{t+k}}{z_{t+k} F'(n_{i,t+k})} \right] = 0. \tag{13}$$

Again express the first-order condition (13) as

$$x_{1,t} - x_{2,t} = 0$$

where

$$x_{1,t} \equiv E_t \sum_{k=0}^{\infty} (\beta\theta_p)^k \frac{\lambda_{t+k}}{\lambda_t} \left(\frac{X_{t,t+k}\tilde{P}_t}{P_{t+k}} \right)^{-\epsilon_p} c_{t+k} \frac{X_{t,t+k}\tilde{P}_t}{P_{t+k}}$$

and

$$x_{2,t} \equiv E_t \sum_{k=0}^{\infty} (\beta\theta_p)^k \frac{\lambda_{t+k}}{\lambda_t} \left(\frac{X_{t,t+k}\tilde{P}_t}{P_{t+k}} \right)^{-\epsilon_p} c_{t+k} \frac{\epsilon_p}{\epsilon_p - 1} \frac{W_{t+k}/P_{t+k}}{z_{t+k} F'(n_{i,t+k})}$$

and then write both $x_{1,t}$ and $x_{2,t}$ recursively. This yields

$$
\begin{aligned}
x_{1,t} &\equiv E_t \sum_{k=0}^{\infty} (\beta\theta_p)^k \frac{\lambda_{t+k}}{\lambda_t} \left(\frac{X_{t,t+k}\tilde{P}_t}{P_{t+k}} \right)^{1-\epsilon_p} c_{t+k} \\
&= \left(\frac{\tilde{P}_t}{P_t} \right)^{1-\epsilon_p} c_t + E_t \sum_{k=1}^{\infty} (\beta\theta_p)^k \frac{\lambda_{t+k}}{\lambda_t} \left(\frac{X_{t,t+k}\tilde{P}_t}{P_{t+k}} \right)^{1-\epsilon_p} c_{t+k} \\
&= \left(\frac{\tilde{P}_t}{P_t} \right)^{1-\epsilon_p} c_t + E_t \sum_{k=1}^{\infty} (\beta\theta_p)^k \frac{\lambda_{t+1}}{\lambda_t} \frac{\lambda_{t+k}}{\lambda_{t+1}} \left(\frac{X_{t,t+k}\tilde{P}_t}{X_{t+1,t+k}\tilde{P}_{t+1}} \frac{X_{t+1,t+k}\tilde{P}_{t+1}}{P_{t+k}} \right)^{1-\epsilon_p} c_{t+k} \\
&= \left(\frac{\tilde{P}_t}{P_t} \right)^{1-\epsilon_p} c_t + \beta\theta_p E_t \frac{\lambda_{t+1}}{\lambda_t} \left(\frac{\tilde{P}_t}{\tilde{P}_{t+1}} ((1+\pi^*)^{\chi^*}(1+\pi_t)^{\chi^p}) \right)^{1-\epsilon_p} x_{1,t+1} \\
&= \tilde{p}_t^{1-\epsilon_p} c_t + \beta\theta_p E_t \frac{\lambda_{t+1}}{\lambda_t} \left(\frac{((1+\pi^*)^{\chi^*}(1+\pi_t)^{\chi^p})\tilde{p}_t}{(1+\pi_{t+1})\tilde{p}_{t+1}} \right)^{1-\epsilon_p} x_{1,t+1}
\end{aligned}
$$

To express $x_{2,t}$ recursively we make use of the assumption that $F(n) = n^\alpha$ so that $F'(n) = \alpha n^{\alpha-1}$. Note that we have used the notation $n_{i,t+k}$ to indicate the number of hours by firm i that last reoptimized its price in period t. To make this clear, let's use the notation $n_{t+k|t}$ for a firm that last optimized in t and $n_{t+k|t+1}$ for a firm that last reoptimized in period $t+1$. With this assumption about $F(\cdot)$, this new notation, and using equation (6) holding with equality, that is, using $z_{t+k}(n_{t+k|t})^\alpha = \left(\frac{X_{t,t+k}\tilde{P}_t}{P_{t+k}}\right)^{-\epsilon_p} c_{t+k}$, we have that

$$
\frac{F'(n_{t+1+h|t+1})}{F'(n_{t+1+h|t})} = \left(\frac{(1+\pi^*)^{\chi^*}(1+\pi_t)^{\chi^p}\tilde{P}_t}{\tilde{P}_{t+1}} \right)^{-\epsilon_p/\alpha+\epsilon_p}
$$

With this notation we can write $x_{2,t+1}$ as follows

$$
x_{2,t+1} = E_{t+1} \sum_{h=0}^{\infty} (\beta\theta_p)^h \frac{\lambda_{t+1+h}}{\lambda_{t+1}} \left(\frac{X_{t+1,t+1+h}\tilde{P}_{t+1}}{P_{t+1+h}} \right)^{-\epsilon_p} c_{t+1+h} \frac{\epsilon_p}{\epsilon_p - 1} \frac{W_{t+1+h}/P_{t+1+h}}{z_{t+1+h}F'(n_{t+1+h|t+1})}
$$

Now write $x_{2,t}$ as

$$
\begin{aligned}
x_{2,t} &= \left(\frac{\tilde{P}_t}{P_t}\right)^{-\epsilon_p} c_t \frac{\epsilon_p}{\epsilon_p - 1}\frac{W_t/P_t}{z_t F'(n_{t|t})} + E_t \sum_{k=1}^{\infty} (\beta\theta_p)^k \frac{\lambda_{t+k}}{\lambda_t}\left(\frac{X_{t,t+k}\tilde{P}_t}{P_{t+k}}\right)^{-\epsilon_p} c_{t+k}\frac{\epsilon_p}{\epsilon_p - 1}\frac{W_{t+k}/P_{t+k}}{z_{t+k}F'(n_{t+k|t})} \\
&= \left(\frac{\tilde{P}_t}{P_t}\right)^{-\epsilon_p} c_t \frac{\epsilon_p}{\epsilon_p - 1}\frac{W_t/P_t}{z_t F'(n_{t|t})} + (\beta\theta_p) E_t \frac{\lambda_{t+1}}{\lambda_t}\sum_{h=0}^{\infty}\left(\frac{X_{t,t+1+h}\tilde{P}_t}{X_{t+1,t+1+h}\tilde{P}_{t+1}}\right)^{-\epsilon_p}\frac{F'(n_{t+1+h|t+1})}{F'(n_{t+1+h|t})} \\
&\quad (\beta\theta_p)^h \frac{\lambda_{t+1+h}}{\lambda_{t+1}}\left(\frac{X_{t+1,t+1+h}\tilde{P}_{t+1}}{P_{t+1+h}}\right)^{-\epsilon_p} c_{t+1+h}\frac{\epsilon_p}{\epsilon_p - 1}\frac{W_{t+1+h}/P_{t+1+h}}{z_{t+1+h}F'(n_{t+1+h|t+1})} \\
&= \left(\frac{\tilde{P}_t}{P_t}\right)^{-\epsilon_p} c_t \frac{\epsilon_p}{\epsilon_p - 1}\frac{W_t/P_t}{z_t F'(n_{t|t})} + \beta\theta_p E_t \frac{\lambda_{t+1}}{\lambda_t}\left(\frac{(1+\pi^*)^{\chi^*}(1+\pi_t)^{\chi^p}\tilde{P}_t}{\tilde{P}_{t+1}}\right)^{-\epsilon_p/\alpha} x_{2,t+1} \\
&= \tilde{p}_t^{-\epsilon_p} c_t \frac{\epsilon_p}{\epsilon_p - 1}\frac{W_t/P_t}{z_t F'(n_{t|t})} + \beta\theta_p E_t \frac{\lambda_{t+1}}{\lambda_t}\left(\frac{(1+\pi^*)^{\chi^*}(1+\pi_t)^{\chi^p}\tilde{p}_t}{(1+\pi_{t+1})\tilde{p}_{t+1}}\right)^{-\epsilon_p/\alpha} x_{2,t+1}
\end{aligned}
$$

In sum, we can express the first-order condition of the firm, equation (7), recursively and equation (6) holding with equality with the following four equations:

$$
\begin{aligned}
x_{1,t} &= x_{2,t} && (14) \\
x_{1,t} &= \tilde{p}_t^{1-\epsilon_p} c_t + \beta\theta_p E_t \frac{\lambda_{t+1}}{\lambda_t}\left(\frac{(1+\pi^*)^{\chi^*}(1+\pi_t)^{\chi^p}\tilde{p}_t}{(1+\pi_{t+1})\tilde{p}_{t+1}}\right)^{1-\epsilon_p} x_{1,t+1} && (15) \\
x_{2,t} &= \tilde{p}_t^{-\epsilon_p} c_t \frac{\epsilon_p}{\epsilon_p - 1}\frac{w_t}{z_t F'(\tilde{n}_t)} + \beta\theta_p E_t \frac{\lambda_{t+1}}{\lambda_t}\left(\frac{(1+\pi^*)^{\chi^*}(1+\pi_t)^{\chi^p}\tilde{p}_t}{(1+\pi_{t+1})\tilde{p}_{t+1}}\right)^{\frac{-\epsilon_p}{\alpha}} x_{2,t+1} && (16) \\
z_t F(\tilde{n}_t) &= \tilde{p}_t^{-\epsilon_p} c_t && (17)
\end{aligned}
$$

From the definition of the price index, equation (4), we now have

$$
\begin{aligned}
P_t^{1-\epsilon_p} &= \int_0^1 P_{it}^{1-\epsilon_p} di \\
P_t^{1-\epsilon_p} &= \int_{\theta_p} \left((1+\pi^*)^{\chi^*}(1+\pi_{t-1})^{\chi^p} P_{it-1}\right)^{1-\epsilon_p} di + \int_{1-\theta_p} \tilde{P}_t^{1-\epsilon_p} di \\
P_t^{1-\epsilon_p} &= \theta_p \left((1+\pi^*)^{\chi^*}(1+\pi_{t-1})^{\chi^p} P_{t-1}\right)^{1-\epsilon_p} + (1-\theta_p)\tilde{P}_t^{1-\epsilon_p} \\
1 &= \theta_p \left((1+\pi^*)^{\chi^*}(1+\pi_{t-1})^{\chi^p}/(1+\pi_t)\right)^{1-\epsilon_p} + (1-\theta_p)\tilde{p}_t^{1-\epsilon_p}
\end{aligned}
$$

1.4 Unions

Unions are the monopolistic supplier of labor of variety j, $h_{j,t}$. They set the nominal wage rate for each variety of labor j, denoted W_{jt} for $j \in [0,1]$, to maximize the utility of the household.

Let h_t denote labor used by firms. Assume that labor used by firms is an aggregate of all varieties of labor:

$$h_t = \left(\int_0^1 h_{j,t}^{1-\frac{1}{\epsilon_w}} dj \right)^{\frac{1}{1-\frac{1}{\epsilon_w}}}$$

The cost minimizing way to assemble labor implies a demand for each variety of labor of the form

$$h_{j,t} = \left(\frac{W_{j,t}}{W_t} \right)^{-\epsilon_w} h_t \tag{18}$$

which is the labor demand faced by union j. At posted wages W_{jt} workers must satisfy demand. The variable W_t denotes the nominal wage of one unit of h_t and is given by

$$W_t \equiv \left(\int_0^1 W_{j,t}^{1-\epsilon_w} dj \right)^{\frac{1}{1-\epsilon_w}} \tag{19}$$

Union j takes as given all variables entering the household's problem other than $W_{j,t}$ and $h_{j,t}$. The relevant parts for union j of the Lagrangian of the household's utility maximization problem are

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \left\{ -e^{\xi_t} V(h_{j,t}) + \lambda_t \frac{W_{j,t}}{P_t} \left(\frac{W_{j,t}}{W_t} \right)^{-\epsilon_w} h_t + \frac{\lambda_t W_t / P_t}{\mu_{j,t}} \left[h_{j,t} - \left(\frac{W_{j,t}}{W_t} \right)^{-\epsilon_w} h_t \right] \right\},$$

where $\frac{\lambda_t W_t / P_t}{\mu_{j,t}}$ is the Lagrange multiplier on the constraint (18). The first-order condition with respect to $h_{j,t}$ is:

$$e^{\xi_t} V'(h_{j,t}) = \frac{\lambda_t W_t / P_t}{\mu_{j,t}}. \tag{20}$$

Nominal wages are sticky. Each period the union can reoptimize W_{jt} with probability $1-\theta_w$ and must charge the same wage as in the previous period with probability θ_w. Let the nominal wage the union chooses when in period t it can reoptimize it be denoted $\tilde{W}_t$. The union picks $\tilde{W}_t$ to maximize

$$E_t \sum_{k=0}^{\infty} (\beta\theta_w)^k \left\{ \lambda_{t+k} \frac{\tilde{W}_t}{P_{t+k}} \left(\frac{\tilde{W}_t}{W_{t+k}} \right)^{-\epsilon_w} h_{t+k} - \frac{\lambda_{t+k} W_{t+k} / P_{t+k}}{\mu_{j,t+k}} \left[\left(\frac{\tilde{W}_t}{W_{t+k}} \right)^{-\epsilon_w} h_{t+k} \right] \right\},$$

The associated first-order condition is

$$E_t \sum_{k=0}^{\infty} (\beta\theta_w)^k \left\{ \frac{\epsilon_w - 1}{\epsilon_w} \lambda_{t+k} \frac{\tilde{W}_t}{P_{t+k}} \left(\frac{\tilde{W}_t}{W_{t+k}} \right)^{-\epsilon_w} h_{t+k} - \frac{\lambda_{t+k} W_{t+k} / P_{t+k}}{\mu_{j,t+k}} \left[\left(\frac{\tilde{W}_t}{W_{t+k}} \right)^{-\epsilon_w} h_{t+k} \right] \right\} = 0.$$

Let

$$f_{1,t} \quad = \quad E_t \sum_{k=0}^{\infty} (\beta\theta_w)^k \left\{ \frac{\epsilon_w - 1}{\epsilon_w} \lambda_{t+k} \frac{\tilde{W}_t}{P_{t+k}} \left(\frac{\tilde{W}_t}{W_{t+k}} \right)^{-\epsilon_w} h_{t+k} \right\}$$

and

$$f_{2,t} = E_t \sum_{k=0}^{\infty} (\beta\theta_w)^k \left\{ \frac{\lambda_{t+k} W_{t+k}/P_{t+k}}{\mu_{j,t+k}} \left[\left(\frac{\tilde{W}_t}{W_{t+k}} \right)^{-\epsilon_w} h_{t+k} \right] \right\}$$

Then the first-order condition can be written as

$$f_{1,t} = f_{2,t}.$$

Next express $f_{1,t}$ and $f_{2,t}$ recursively.

$$\begin{aligned}
f_{1,t} &= E_t \sum_{k=0}^{\infty} (\beta\theta_w)^k \left\{ \frac{\epsilon_w - 1}{\epsilon_w} \lambda_{t+k} \frac{\tilde{W}_t}{P_{t+k}} \left(\frac{\tilde{W}_t}{W_{t+k}} \right)^{-\epsilon_w} h_{t+k} \right\} \\
&= \frac{\epsilon_w - 1}{\epsilon_w} \lambda_t \frac{\tilde{W}_t}{P_t} \left(\frac{\tilde{W}_t}{W_t} \right)^{-\epsilon_w} h_t + E_t \sum_{k=1}^{\infty} (\beta\theta_w)^k \left\{ \frac{\epsilon_w - 1}{\epsilon_w} \lambda_{t+k} \frac{\tilde{W}_t}{P_{t+k}} \left(\frac{\tilde{W}_t}{W_{t+k}} \right)^{-\epsilon_w} h_{t+k} \right\} \\
&= \frac{\epsilon_w - 1}{\epsilon_w} \lambda_t \frac{\tilde{W}_t}{P_t} \left(\frac{\tilde{W}_t}{W_t} \right)^{-\epsilon_w} h_t + \beta\theta_w E_t \sum_{h=0}^{\infty} \frac{\tilde{W}_t}{\tilde{W}_{t+1}} \left(\frac{\tilde{W}_t}{\tilde{W}_{t+1}} \right)^{-\epsilon_w} \\
&\quad (\beta\theta_w)^h \left\{ \frac{\epsilon_w - 1}{\epsilon_w} \lambda_{t+1+h} \frac{\tilde{W}_{t+1}}{P_{t+1+h}} \left(\frac{\tilde{W}_{t+1}}{W_{t+1+h}} \right)^{-\epsilon_w} h_{t+1+h} \right\} \\
&= \frac{\epsilon_w - 1}{\epsilon_w} \lambda_t \frac{\tilde{W}_t}{P_t} \left(\frac{\tilde{W}_t}{W_t} \right)^{-\epsilon_w} h_t + \beta\theta_w E_t \left(\frac{\tilde{W}_t}{\tilde{W}_{t+1}} \right)^{1-\epsilon_w} f_{1,t+1}
\end{aligned}$$

To express $f_{2,t}$ recursively use the following intermediate results. Use equation (20) to replace $\lambda_{t+k} W_{t+k}/P_{t+k}/\mu_{j,t+k}$ with $e^{\xi_{t+k}} V'(h_{j,t+k})$, use the assumption that $V'(x) = x^{\varphi}$, and replace $h_{j,t+k}$ with (18). This yields for any $k \geq 1$

$$\lambda_{t+k} W_{t+k}/P_{t+k}/\mu_{j,t+k} = e^{\xi_{t+k}} \left[\left(\frac{\tilde{W}_t}{W_{t+k}} \right)^{-\epsilon_w} h_{t+k} \right]^{\varphi}$$

With these intermediate results in hand, we can express $f_{2,t}$ recursively as follows

$$\begin{aligned}
f_{2,t} &= E_t \sum_{k=0}^{\infty} (\beta\theta_w)^k \left\{ \frac{\lambda_{t+k} W_{t+k}/P_{t+k}}{\mu_{j,t+k}} \left[\left(\frac{\tilde{W}_t}{W_{t+k}} \right)^{-\epsilon_w} h_{t+k} \right] \right\} \\
&= e^{\xi_t} \left[\left(\frac{\tilde{W}_t}{W_t} \right)^{-\epsilon_w} h_t \right]^{\varphi+1} + E_t \sum_{k=1}^{\infty} (\beta\theta_w)^k \left\{ e^{\xi_{t+k}} \left[\left(\frac{\tilde{W}_t}{W_{t+k}} \right)^{-\epsilon_w} h_{t+k} \right]^{\varphi+1} \right\} \\
&= e^{\xi_t} \left[\left(\frac{\tilde{W}_t}{W_t} \right)^{-\epsilon_w} h_t \right]^{\varphi+1} + \beta\theta_w E_t \sum_{h=0}^{\infty} (\beta\theta_w)^h \left[\left(\frac{\tilde{W}_t}{\tilde{W}_{t+1}} \right)^{-\epsilon_w} \right]^{\varphi+1} \left\{ e^{\xi_{t+1+h}} \left[\left(\frac{\tilde{W}_{t+1}}{W_{t+1+h}} \right)^{-\epsilon_w} h_{t+1+h} \right]^{\varphi+1} \right\} \\
&= e^{\xi_t} \left[\left(\frac{\tilde{W}_t}{W_t} \right)^{-\epsilon_w} h_t \right]^{\varphi+1} + \beta\theta_w E_t \left[\left(\frac{\tilde{W}_t}{\tilde{W}_{t+1}} \right)^{-\epsilon_w} \right]^{\varphi+1} f_{2,t+1}
\end{aligned}$$

Let

$$w_t \equiv \frac{W_t}{P_t},$$

$$\tilde{w}_t \equiv \frac{\tilde{W}_t}{W_t}$$

and

$$1 + \pi_t^W \equiv \frac{W_t}{W_{t-1}}$$

we can express the first-order condition recursively in the following 3 equations

$$\begin{aligned} f_{1t} &= f_{2,t} && (21) \\ f_{1t} &= \frac{\epsilon_w - 1}{\epsilon_w} \lambda_t \tilde{w}_t^{1-\epsilon_w} w_t h_t + \beta\theta_w E_t \left(\frac{\tilde{w}_t}{(1+\pi_{t+1}^W)\tilde{w}_{t+1}} \right)^{1-\epsilon_w} f_{1,t+1} && (22) \\ f_{2,t} &= e^{\xi_t} \left[\tilde{w}_t^{-\epsilon_w} h_t \right]^{\varphi+1} + \beta\theta_w E_t \left[\left(\frac{\tilde{w}_t}{(1+\pi_{t+1}^W)\tilde{w}_{t+1}} \right)^{-\epsilon_w} \right]^{\varphi+1} f_{2,t+1} && (23) \end{aligned}$$

From the definition of the wage index (19), we have

$$\begin{aligned} W_t^{1-\epsilon_w} &= \int_0^1 W_{j,t}^{1-\epsilon_w} dj \\ W_t^{1-\epsilon_w} &= \int_{\theta_w} W_{j,t-1}^{1-\epsilon_w} dj + \int_{1-\theta_w} \tilde{W}_t^{1-\epsilon_w} dj \\ W_t^{1-\epsilon_w} &= \theta_w W_{t-1}^{1-\epsilon_w} + (1-\theta_w)\tilde{W}_t^{1-\epsilon_w} \\ 1 &= \theta_w (1+\pi_t^W)^{\epsilon_w - 1} + (1-\theta_w)\tilde{w}_t^{1-\epsilon_w} && (24) \end{aligned}$$

1.5 Unions with Indexation to Price Inflation

Nominal wages are sticky. Each period the union can reoptimize W_{jt} with probability $1-\theta_w$. When the union cannot reoptimize (which occurs with probability θ_w), then the nominal wage adjusts as follows:

$$W_{jt} = (1+\pi^*)^{\nu^*}(1+\pi_{t-1})^{\nu^p} W_{j,t-1},$$

where $\nu^*, \nu^p \geq 0$ are parameters controlling the degree of indexation. This specification nests the case of no indexation. In particular, if $\nu^* = \nu^p = 0$, then the union has to charge the same nominal wage as in the previous period. If $\nu^* + \nu^p = 1$, then there is full wage indexation in the long run.

Let the nominal wage the union chooses when in period t it can reoptimize it be denoted $\tilde{W}_t$ and let

$$\Upsilon_{t,t+k} \equiv \Pi_{j=0}^{k-1}(1+\pi^*)^{\nu^*}(1+\pi_{t+j})^{\nu^p},$$

with

$$\Upsilon_{t,t} = 1.$$

So, if a union last got to reoptimize the wage in period t, then in period $t+k$ the wage it charges is equal to

$$\Upsilon_{t,t+k}\tilde{W}_t$$

In period t the union picks $\tilde{W}_t$ to maximize

$$E_t \sum_{k=0}^{\infty} (\beta\theta_w)^k \left\{ \lambda_{t+k} \frac{Y_{t,t+k}\tilde{W}_t}{P_{t+k}} \left(\frac{Y_{t,t+k}\tilde{W}_t}{W_{t+k}} \right)^{-\epsilon_w} h_{t+k} - \frac{\lambda_{t+k} W_{t+k}/P_{t+k}}{\mu_{j,t+k}} \left[\left(\frac{Y_{t,t+k}\tilde{W}_t}{W_{t+k}} \right)^{-\epsilon_w} h_{t+k} \right] \right\}.$$

The associated first-order condition is

$$E_t \sum_{k=0}^{\infty} (\beta\theta_w)^k \left\{ \frac{\epsilon_w - 1}{\epsilon_w} \lambda_{t+k} \frac{Y_{t,t+k}\tilde{W}_t}{P_{t+k}} \left(\frac{Y_{t,t+k}\tilde{W}_t}{W_{t+k}} \right)^{-\epsilon_w} h_{t+k} - \frac{\lambda_{t+k} W_{t+k}/P_{t+k}}{\mu_{j,t+k}} \left[\left(\frac{Y_{t,t+k}\tilde{W}_t}{W_{t+k}} \right)^{-\epsilon_w} h_{t+k} \right] \right\} = 0.$$

Let

$$f_{1,t} = E_t \sum_{k=0}^{\infty} (\beta\theta_w)^k \left\{ \frac{\epsilon_w - 1}{\epsilon_w} \lambda_{t+k} \frac{Y_{t,t+k}\tilde{W}_t}{P_{t+k}} \left(\frac{Y_{t,t+k}\tilde{W}_t}{W_{t+k}} \right)^{-\epsilon_w} h_{t+k} \right\}$$

and

$$f_{2,t} = E_t \sum_{k=0}^{\infty} (\beta\theta_w)^k \left\{ \frac{\lambda_{t+k} W_{t+k}/P_{t+k}}{\mu_{j,t+k}} \left[\left(\frac{Y_{t,t+k}\tilde{W}_t}{W_{t+k}} \right)^{-\epsilon_w} h_{t+k} \right] \right\}.$$

Then the first-order condition can be written as

$$f_{1,t} = f_{2,t}.$$

Next express $f_{1,t}$ and $f_{2,t}$ recursively.

$$\begin{aligned}
f_{1,t} &= E_t \sum_{k=0}^{\infty} (\beta\theta_w)^k \left\{ \frac{\epsilon_w - 1}{\epsilon_w} \lambda_{t+k} \frac{Y_{t,t+k}\tilde{W}_t}{P_{t+k}} \left(\frac{Y_{t,t+k}\tilde{W}_t}{W_{t+k}} \right)^{-\epsilon_w} h_{t+k} \right\} \\
&= \frac{\epsilon_w - 1}{\epsilon_w} \lambda_t \frac{\tilde{W}_t}{P_t} \left(\frac{\tilde{W}_t}{W_t} \right)^{-\epsilon_w} h_t + E_t \sum_{k=1}^{\infty} (\beta\theta_w)^k \left\{ \frac{\epsilon_w - 1}{\epsilon_w} \lambda_{t+k} \frac{Y_{t,t+k}\tilde{W}_t}{P_{t+k}} \left(\frac{Y_{t,t+k}\tilde{W}_t}{W_{t+k}} \right)^{-\epsilon_w} h_{t+k} \right\} \\
&= \frac{\epsilon_w - 1}{\epsilon_w} \lambda_t \frac{\tilde{W}_t}{P_t} \left(\frac{\tilde{W}_t}{W_t} \right)^{-\epsilon_w} h_t + \beta\theta_w E_t \sum_{h=0}^{\infty} \left(\frac{Y_{t,t+1+h}\tilde{W}_t}{Y_{t+1,t+1+h}\tilde{W}_{t+1}} \right)^{1-\epsilon_w} \\
&\quad (\beta\theta_w)^h \left\{ \frac{\epsilon_w - 1}{\epsilon_w} \lambda_{t+1+h} \frac{Y_{t+1,t+1+h}\tilde{W}_{t+1}}{P_{t+1+h}} \left(\frac{Y_{t+1,t+1+h}\tilde{W}_{t+1}}{W_{t+1+h}} \right)^{-\epsilon_w} h_{t+1+h} \right\} \\
&= \frac{\epsilon_w - 1}{\epsilon_w} \lambda_t \frac{\tilde{W}_t}{P_t} \left(\frac{\tilde{W}_t}{W_t} \right)^{-\epsilon_w} h_t + \beta\theta_w E_t \left(\frac{(1+\pi^*)^{\nu^*}(1+\pi_t)^{\nu^p}\tilde{W}_t}{\tilde{W}_{t+1}} \right)^{1-\epsilon_w} f_{1,t+1}
\end{aligned}$$

To express $f_{2,t}$ recursively use the following intermediate results. Use equation (20) to replace $\lambda_{t+k}W_{t+k}/P_{t+k}/\mu_{j,t+k}$ with $e^{\xi_{t+k}}V'(h_{j,t+k})$, use the assumption that $V'(x) = x^{\varphi}$, and replace $h_{j,t+k}$ with (18). This yields for any $k \geq 0$

$$\lambda_{t+k} W_{t+k}/P_{t+k}/\mu_{j,t+k} = e^{\xi_{t+k}} \left[\left(\frac{Y_{t,t+k}\tilde{W}_t}{W_{t+k}} \right)^{-\epsilon_w} h_{t+k} \right]^{\varphi}.$$

With these intermediate results in hand, we can express $f_{2,t}$ recursively as follows

$$
\begin{aligned}
f_{2,t} &= E_t \sum_{k=0}^{\infty} (\beta\theta_w)^k \left\{ \frac{\lambda_{t+k} W_{t+k}/P_{t+k}}{\mu_{j,t+k}} \left[\left(\frac{Y_{t,t+k}\tilde{W}_t}{W_{t+k}} \right)^{-\epsilon_w} h_{t+k} \right] \right\} \\
&= e^{\xi_t} \left[\left(\frac{\tilde{W}_t}{W_t} \right)^{-\epsilon_w} h_t \right]^{\varphi+1} + E_t \sum_{k=1}^{\infty} (\beta\theta_w)^k \left\{ e^{\xi_{t+k}} \left[\left(\frac{Y_{t,t+k}\tilde{W}_t}{W_{t+k}} \right)^{-\epsilon_w} h_{t+k} \right]^{\varphi+1} \right\} \\
&= e^{\xi_t} \left[\left(\frac{\tilde{W}_t}{W_t} \right)^{-\epsilon_w} h_t \right]^{\varphi+1} + \beta\theta_w E_t \sum_{h=0}^{\infty} (\beta\theta_w)^h \left[\left(\frac{Y_{t,t+1+h}\tilde{W}_t}{Y_{t+1,t+1+h}\tilde{W}_{t+1}} \right)^{-\epsilon_w} \right]^{\varphi+1} \\
&\quad \left\{ e^{\xi_{t+1+h}} \left[\left(\frac{Y_{t+1,t+1+h}\tilde{W}_{t+1}}{W_{t+1+h}} \right)^{-\epsilon_w} h_{t+1+h} \right]^{\varphi+1} \right\} \\
&= e^{\xi_t} \left[\left(\frac{\tilde{W}_t}{W_t} \right)^{-\epsilon_w} h_t \right]^{\varphi+1} + \beta\theta_w E_t \left[\left(\frac{(1+\pi^*)^{\nu^*}(1+\pi_t)^{\nu^p}\tilde{W}_t}{\tilde{W}_{t+1}} \right)^{-\epsilon_w} \right]^{\varphi+1} f_{2,t+1}
\end{aligned}
$$

Recalling the definitions,

$$w_t \equiv \frac{W_t}{P_t},$$

$$\tilde{w}_t \equiv \frac{\tilde{W}_t}{W_t}$$

and

$$1 + \pi_t^W \equiv \frac{W_t}{W_{t-1}},$$

we can express the first-order condition for the case of indexation recursively in the following 3 equations

$$f_{1t} = f_{2,t} \tag{25}$$

$$f_{1t} = \frac{\epsilon_w - 1}{\epsilon_w} \lambda_t \tilde{w}_t^{1-\epsilon_w} w_t h_t + \beta\theta_w E_t \left(\frac{(1+\pi^*)^{\nu^*}(1+\pi_t)^{\nu^p}\tilde{w}_t}{(1+\pi_{t+1}^W)\tilde{w}_{t+1}} \right)^{1-\epsilon_w} f_{1,t+1} \tag{26}$$

$$f_{2,t} = e^{\xi_t} \left[\tilde{w}_t^{-\epsilon_w} h_t \right]^{\varphi+1} + \beta\theta_w E_t \left[\left(\frac{(1+\pi^*)^{\nu^*}(1+\pi_t)^{\nu^p}\tilde{w}_t}{(1+\pi_{t+1}^W)\tilde{w}_{t+1}} \right)^{-\epsilon_w} \right]^{\varphi+1} f_{2,t+1} \tag{27}$$

Again, use the definition of the wage index (19), we obtain

$$
\begin{aligned}
W_t^{1-\epsilon_w} &= \int_0^1 W_{j,t}^{1-\epsilon_w} dj \\
W_t^{1-\epsilon_w} &= \int_{\theta_w} \left[(1+\pi^*)^{\nu^*}(1+\pi_{t-1})^{\nu^p} W_{j,t-1} \right]^{1-\epsilon_w} dj + \int_{1-\theta_w} \tilde{W}_t^{1-\epsilon_w} dj \\
W_t^{1-\epsilon_w} &= \theta_w \left[(1+\pi^*)^{\nu^*}(1+\pi_{t-1})^{\nu^p} \right]^{1-\epsilon_w} W_{t-1}^{1-\epsilon_w} + (1-\theta_w)\tilde{W}_t^{1-\epsilon_w} \\
1 &= \theta_w \left[(1+\pi^*)^{\nu^*}(1+\pi_{t-1})^{\nu^p} \right]^{1-\epsilon_w} (1+\pi_t^W)^{\epsilon_w - 1} + (1-\theta_w)\tilde{w}_t^{1-\epsilon_w} \qquad (28)
\end{aligned}
$$

1.6 Aggregation and Market Clearing

Hours hired by firms must add up to the labor index, that is,

$$h_t = \int_0^1 n_{i,t} di. \tag{29}$$

Using the assumed functional form for $F(\cdot)$ and the fact that in equilibrium (6) holds with equality we have

$$z_t n_{it}^{\alpha} = \left(\frac{P_{i,t}}{P_t}\right)^{-\epsilon_p} c_t$$

Solve this expresssion for $n_{i,t}$ to obtain

$$n_{i,t} = \left(\frac{\left(\frac{P_{it}}{P_t}\right)^{-\epsilon_p} c_t}{z_t}\right)^{\frac{1}{\alpha}}$$

Integrate over i

$$\int_0^1 n_{i,t} di = \int_0^1 \left(\frac{\left(\frac{P_{it}}{P_t}\right)^{-\epsilon_p} c_t}{z_t}\right)^{\frac{1}{\alpha}} di$$

Combine with (29)

$$h_t = \int_0^1 \left(\frac{\left(\frac{P_{it}}{P_t}\right)^{-\epsilon_p} c_t}{z_t}\right)^{\frac{1}{\alpha}} di$$

Letting

$$s_t \equiv \int_0^1 \left(\frac{P_{it}}{P_t}\right)^{-\frac{\epsilon_p}{\alpha}} di$$

and rearranging yields the aggregate resource constraint

$$c_t = z_t h_t^{\alpha} s_t^{-\alpha}. \tag{30}$$

Write s_t recursively,

$$\begin{aligned}
s_t &= \int_0^1 \left(\frac{P_{it}}{P_t}\right)^{-\frac{\epsilon_p}{\alpha}} di \\
&= \int_{\theta_p} \left(\frac{P_{it-1}}{P_t}\right)^{-\frac{\epsilon_p}{\alpha}} di + \int_{1-\theta_p} \tilde{p}_t^{-\frac{\epsilon_p}{\alpha}} di \\
&= \int_{\theta_p} \left(\frac{P_{it-1}/P_{t-1}}{P_t/P_{t-1}}\right)^{-\frac{\epsilon_p}{\alpha}} di + (1-\theta_p)\tilde{p}_t^{-\frac{\epsilon_p}{\alpha}} \\
&= \theta_p s_{t-1}(1+\pi_t)^{\frac{\epsilon_p}{\alpha}} + (1-\theta_p)\tilde{p}_t^{-\frac{\epsilon_p}{\alpha}}
\end{aligned}$$

That is, we have

$$s_t = \theta_p s_{t-1}(1+\pi_t)^{\frac{\epsilon_p}{\alpha}} + (1-\theta_p)\tilde{p}_t^{-\frac{\epsilon_p}{\alpha}} \tag{31}$$

Finally, let y_t denote the supply of final goods, which is defined to be the same as absorption, that is,

$$y_t \equiv c_t.$$

In the special case, in which in the non-stochastic steady state $\pi = 0$, we have that $\tilde{p} = 1$. And then we can infer from the above expression for s_t and y_t, that that $s = 1$ and that $y = zh^{\alpha}$.

1.7 Aggregation and Market Clearing with Indexation

Equilibrium conditions (29), (30) unchanged. The definition of s_t is also unchanged. But writing it recursively now has to be adjusted for indexation.

Write s_t recursively,

$$\begin{aligned}
s_t &= \int_0^1 \left(\frac{P_{it}}{P_t}\right)^{-\frac{\epsilon_p}{\alpha}} di \\
&= \int_{\theta_p} \left(\frac{P_{it}}{P_t}\right)^{-\frac{\epsilon_p}{\alpha}} di + \int_{1-\theta_p} \tilde{p}_t^{-\frac{\epsilon_p}{\alpha}} di \\
&= \int_{\theta_p} \left(\frac{(1+\pi^*)^{\chi^*}(1+\pi_{t-1})^{\chi^p} P_{it-1}/P_{t-1}}{P_t/P_{t-1}}\right)^{-\frac{\epsilon_p}{\alpha}} di + (1-\theta_p)\tilde{p}_t^{-\frac{\epsilon_p}{\alpha}} \\
&= \left[\frac{(1+\pi^*)^{\chi^*}(1+\pi_{t-1})^{\chi^p}}{1+\pi_t}\right]^{-\frac{\epsilon_p}{\alpha}} \int_{\theta_p} \left(\frac{P_{it-1}}{P_{t-1}}\right)^{-\frac{\epsilon_p}{\alpha}} di + (1-\theta_p)\tilde{p}_t^{-\frac{\epsilon_p}{\alpha}} \\
&= \left[\frac{(1+\pi^*)^{\chi^*}(1+\pi_{t-1})^{\chi^p}}{1+\pi_t}\right]^{-\frac{\epsilon_p}{\alpha}} \theta_p s_{t-1} + (1-\theta_p)\tilde{p}_t^{-\frac{\epsilon_p}{\alpha}}
\end{aligned}$$

1.8 Monetary Policy

Monetary policy is the same as in the HDNWR model; the central bank sets the nominal interest rate according to a Taylor rule of the form

$$1 + i_t = \frac{1+\pi^*}{\beta}\left(\frac{1+\pi_t}{1+\pi^*}\right)^{\alpha_\pi}\left(\frac{y_t}{y}\right)^{\alpha_y}\mu_t, \tag{32}$$

where π^* denotes the central bank's inflation target, y is the non-stochastic steady state value of output, $\beta\alpha_\pi > 1$ is a parameter, and μ_t is an exogenous and stochastic monetary shock.

1.9 Complete Set of Equilibrium Conditions

Finally, we can define an equilibrium. For convenience we will repeat all equilibrium conditions. An equilibrium is a set of stochastic processes, $\{i_t,\, c_t,\, \pi_t,\, \pi_t^W,\, \lambda_t,\, \tilde{p}_t,\, x_{1t},\, x_{2t},\, w_t,\, \tilde{n}_t,\, f_{1t},\, f_{2t},\, \tilde{w}_t,\, h_t,\, s_t\}$ satisfying

$$e^{\xi_t} U'(c_t) = \beta(1+i_t) E_t \left[e^{\xi_{t+1}} \frac{U'(c_{t+1})}{1+\pi_{t+1}} \right] \tag{33}$$

$$\lambda_t = e^{\xi_t} U'(c_t) \tag{34}$$

$$x_{1,t} = x_{2,t} \tag{35}$$

$$x_{1,t} = \tilde{p}_t^{1-\epsilon_p} c_t + \beta \theta_p E_t \frac{\lambda_{t+1}}{\lambda_t} \left(\frac{\tilde{p}_t}{(1+\pi_{t+1})\tilde{p}_{t+1}} \right)^{1-\epsilon_p} x_{1,t+1} \tag{36}$$

$$x_{2,t} = \tilde{p}_t^{-\epsilon_p} c_t \frac{\epsilon_p}{\epsilon_p - 1} \frac{w_t}{z_t F'(\tilde{n}_t)} + \beta \theta_p E_t \frac{\lambda_{t+1}}{\lambda_t} \left(\frac{\tilde{p}_t}{(1+\pi_{t+1})\tilde{p}_{t+1}} \right)^{\frac{-\epsilon_p}{\alpha}} x_{2,t+1} \tag{37}$$

$$z_t F(\tilde{n}_t) = \tilde{p}_t^{-\epsilon_p} c_t \tag{38}$$

$$1 = \theta_p (1+\pi_t)^{\epsilon_p - 1} + (1-\theta_p) \tilde{p}_t^{1-\epsilon_p} \tag{39}$$

$$1+\pi_t = (1+\pi_t^W) \frac{w_{t-1}}{w_t} \tag{40}$$

$$f_{1t} = f_{2,t} \tag{41}$$

$$f_{1t} = \frac{\epsilon_w - 1}{\epsilon_w} \lambda_t \tilde{w}_t^{1-\epsilon_w} w_t h_t + \beta \theta_w E_t \left(\frac{\tilde{w}_t}{(1+\pi_{t+1}^W)\tilde{w}_{t+1}} \right)^{1-\epsilon_w} f_{1,t+1} \tag{42}$$

$$f_{2,t} = e^{\xi_t} \left[\tilde{w}_t^{-\epsilon_w} h_t \right]^{\varphi+1} + \beta \theta_w E_t \left[\left(\frac{\tilde{w}_t}{(1+\pi_{t+1}^W)\tilde{w}_{t+1}} \right)^{-\epsilon_w} \right]^{\varphi+1} f_{2,t+1} \tag{43}$$

$$1 = \theta_w (1+\pi_t^W)^{\epsilon_w - 1} + (1-\theta_w) \tilde{w}_t^{1-\epsilon_w} \tag{44}$$

$$c_t = z_t h_t^{\alpha} s_t^{-\alpha} \tag{45}$$

$$s_t = \theta_p s_{t-1} (1+\pi_t)^{\frac{\epsilon_p}{\alpha}} + (1-\theta_p) \tilde{p}_t^{-\frac{\epsilon_p}{\alpha}} \tag{46}$$

$$1+i_t = \frac{1+\pi^*}{\beta} \left(\frac{1+\pi_t}{1+\pi^*} \right)^{\alpha_\pi} \left(\frac{y_t}{y} \right)^{\alpha_y} \mu_t, \tag{47}$$

given initial conditions s_{-1} and w_{-1}, and exogenous processes z_t, ξ_t, and μ_t.

Note that up to a first order approximation s_t only depends on s_{t-1}. Thus, if $s_{-1} = 1$, then up to a first-order approximation $s_t = 1$ for all t and has no impact on the equilibrium dynamics.

1.10 Complete Set of Equilibrium Conditions With Indexation

An equilibrium is a set of stochastic processes, $\{i_t, c_t, \pi_t, \pi_t^W, \lambda_t, \tilde{p}_t, x_{1t}, x_{2t}, w_t, \tilde{n}_t, f_{1t}, f_{2t}, \tilde{w}_t, h_t, s_t\}$ satisfying (33), (34), (35), (38), (40), (41), (45), (47), and

$$x_{1,t} = \tilde{p}_t^{1-\epsilon_p} c_t + \beta\theta_p E_t \frac{\lambda_{t+1}}{\lambda_t} \left(\frac{(1+\pi^*)^{\chi^*}(1+\pi_t)^{\chi^p}\tilde{p}_t}{(1+\pi_{t+1})\tilde{p}_{t+1}} \right)^{1-\epsilon_p} x_{1,t+1} \tag{48}$$

$$x_{2,t} = \tilde{p}_t^{-\epsilon_p} c_t \frac{\epsilon_p}{\epsilon_p - 1} \frac{w_t}{z_t F'(\tilde{n}_t)} + \beta\theta_p E_t \frac{\lambda_{t+1}}{\lambda_t} \left(\frac{(1+\pi^*)^{\chi^*}(1+\pi_t)^{\chi^p}\tilde{p}_t}{(1+\pi_{t+1})\tilde{p}_{t+1}} \right)^{\frac{-\epsilon_p}{\alpha}} x_{2,t+1} \tag{49}$$

$$1 = \left[(1+\pi^*)^{\chi^*}(1+\pi_{t-1})^{\chi^p}\right]^{1-\epsilon_p} \theta_p (1+\pi_t)^{\epsilon_p - 1} + (1-\theta_p)\tilde{p}_t^{1-\epsilon_p} \tag{50}$$

$$f_{1t} = \frac{\epsilon_w - 1}{\epsilon_w} \lambda_t \tilde{w}_t^{1-\epsilon_w} w_t h_t + \beta\theta_w E_t \left(\frac{(1+\pi^*)^{\nu^*}(1+\pi_t)^{\nu^p}\tilde{w}_t}{(1+\pi_{t+1}^W)\tilde{w}_{t+1}} \right)^{1-\epsilon_w} f_{1,t+1} \tag{51}$$

$$f_{2,t} = e^{\xi_t} \left[\tilde{w}_t^{-\epsilon_w} h_t\right]^{\varphi+1} + \beta\theta_w E_t \left[\left(\frac{(1+\pi^*)^{\nu^*}(1+\pi_t)^{\nu^p}\tilde{w}_t}{(1+\pi_{t+1}^W)\tilde{w}_{t+1}} \right)^{-\epsilon_w} \right]^{\varphi+1} f_{2,t+1} \tag{52}$$

$$1 = \left[(1+\pi^*)^{\nu^*}(1+\pi_{t-1})^{\nu^p}\right]^{1-\epsilon_w} \theta_w (1+\pi_t^W)^{\epsilon_w - 1} + (1-\theta_w)\tilde{w}_t^{1-\epsilon_w} \tag{53}$$

$$s_t = \left[(1+\pi^*)^{\chi^*}(1+\pi_{t-1})^{\chi^p}\right]^{-\frac{\epsilon_p}{\alpha}} \theta_p s_{t-1} (1+\pi_t)^{\frac{\epsilon_p}{\alpha}} + (1-\theta_p)\tilde{p}_t^{-\frac{\epsilon_p}{\alpha}}, \tag{54}$$

given initial conditions s_{-1}, w_{-1}, and π_{-1}, and exogenous processes z_t, ξ_t, and μ_t.

Indexation introduces one new lagged endogenous state, namely, π_{t-1} and 4 new parameters: χ^*, χ^p, ν^*, and ν^p, all related to the degree of indexation.

1.11 Unemployment

We define unemployment the same way as in the heterogenous wage rigidity model with endogenous labor supply. Specifically, let u_t denote the unemployment rate. Then u_t is given by

$$u_t = \frac{\int_0^1 (h_{it}^s - h_{it})di}{\int_0^1 h_{it}^s di},$$

where h_{it}^s denotes the quantity of variety i of labor the household would like to work given the wage rate $W_{i,t}$. This labor supply is implicitly given by the following household first-order condition

$$e^{\xi_t} V'(h_{it}^s) = \lambda_t \frac{W_{i,t}}{P_t}$$

which says that the household wishes to supply labor of variety i up to the point where the marginal disutility of doing so is equal to the marginal utility benefit. Using the assumed functional form of V, we have $V'(x) = x^\varphi$. Solving the above expression for h_{it}^s yields

$$h_{it}^s = \left(\frac{\lambda_t \frac{W_t}{P_t}}{e^{\xi_t}} \right)^{\frac{1}{\varphi}} \left(\frac{W_{i,t}}{W_t} \right)^{\frac{1}{\varphi}} \tag{55}$$

Let

$$v_t \equiv \int_0^1 \left(\frac{W_{i,t}}{W_t} \right)^{\frac{1}{\varphi}} di.$$

With this notation in hand total labor supply becomes

$$\int_0^1 h_{it}^s di = \left(\frac{\lambda_t \frac{W_t}{P_t}}{e^{\xi_t}}\right)^{\frac{1}{\varphi}} v_t.$$

The variable v_t can be written recursively as

$$v_t = \left(\frac{A_{w,t}}{1+\pi_t^W}\right)^{\frac{1}{\varphi}} \theta_w v_{t-1} + (1-\theta_w)\tilde{w}_t^{\frac{1}{\varphi}} \tag{56}$$

where

$$A_{w,t} \equiv (1+\pi^*)^{\nu^*}(1+\pi_{t-1})^{\nu^p}$$

is the wage indexation factor in period t.

The quantity of labor of variety i demanded is given by equation (18), which we repeat here for convenience.

$$h_{i,t} = \left(\frac{W_{i,t}}{W_t}\right)^{-\epsilon_w} h_t \tag{18}$$

Taken the integral over all varieties we have

$$\int_0^1 h_{i,t} di = d_t h_t$$

with

$$d_t \equiv \int_0^1 \left(\frac{W_{it}}{W_t}\right)^{-\epsilon_w} di$$

and

$$d_t = \theta_w \left(\frac{A_{w,t}}{1+\pi_t^W}\right)^{-\epsilon_w} d_{t-1} + (1-\theta_w)\tilde{w}_t^{-\epsilon_w}. \tag{57}$$

Finally, we have that

$$u_t = 1 - \frac{d_t h_t}{\left(\frac{\lambda_t}{e^{\xi_t}} w_t\right)^{\frac{1}{\varphi}} v_t} \tag{58}$$

1.11.1 Steady State of Unemployment

$$A_w = (1+\pi^*)^{\nu^*+\nu^p}$$

$$v = \frac{1-\theta_w}{1-\left(\frac{A_w}{1+\pi^W}\right)^{\frac{1}{\varphi}}\theta_w} \tilde{w}^{\frac{1}{\varphi}} \tag{59}$$

$$d = \frac{1-\theta_w}{1-\theta_w\left(\frac{A_w}{1+\pi^W}\right)^{-\epsilon_w}} \tilde{w}^{-\epsilon_w}. \tag{60}$$

Evaluating (58) at the steady state gives

$$u = 1 - \frac{dh}{\left(\frac{\lambda w}{e^{\xi}}\right)^{\frac{1}{\varphi}} v}.$$

Note that under full indexation, that is, when $\nu^* + \nu^p = 1$, $d = v = \tilde{w} = s = 1$. From the derivation of the steady state (below), we see that in that case $h^\varphi = \lambda w \frac{\epsilon_w - 1}{\epsilon_w}$, so that $u = 1 - \left(\frac{\epsilon_w - 1}{\epsilon_w}\right)^{\frac{1}{\varphi}}$.

1.12 Non-stochastic Steady State — without indexation

The calibration follows HDNWR and if a parameter does not have a counterpart in HDNWR, Gali (2015). The time unit is a quarter.

$$\theta_p = 0.75$$

$$\theta_w = 0.75$$

$$\sigma = 2$$

$$\alpha = 0.75$$

$$\beta = 0.99$$

$$\alpha_\pi = 1.5$$

$$\eta_p = 9$$

$$\eta_w = 6$$

$$\varphi = 5$$

The steady state inflation is 2 percent per year

$$\pi = 1.02^{(1/4)} - 1$$

The steady state values of the exogenous shocks are

$$z = 1$$

$$\mu = 1$$

$$\xi = 0$$

Solve (33) for i_t

$$(1 + i) = \frac{(1 + \pi)}{\beta}$$

By (40)

$$\pi^W = \pi$$

By (39)

$$\tilde{p} = \left[\frac{1 - \theta_p(1 + \pi)^{\epsilon_p - 1}}{1 - \theta_p}\right]^{\frac{1}{1 - \epsilon_p}}$$

By (44)

$$\tilde{w} = \left[\frac{1 - \theta_w(1 + \pi)^{\epsilon_w - 1}}{1 - \theta_w}\right]^{\frac{1}{1 - \epsilon_w}}$$

By (46)

$$s = \frac{(1-\theta_p)\tilde{p}^{\frac{-\epsilon_p}{\alpha}}}{1-\theta_p(1+\pi)^{\frac{\epsilon_p}{\alpha}}}$$

By (36)

$$x_1 = \frac{\tilde{p}^{1-\epsilon_p}}{1-\beta\theta_p(1+\pi)^{\epsilon_p-1}}c$$

and by (37)

$$x_2 = \frac{\tilde{p}^{-\epsilon_p}\frac{\epsilon_p}{\epsilon_p-1}}{1-\beta\theta_p(1+\pi)^{\frac{\epsilon_p}{\alpha}}}c\frac{w}{\alpha\tilde{n}^{\alpha-1}}$$

By (35)

$$x_1 = x_2$$

Solve the resulting expression for $\frac{w}{\alpha\tilde{n}^{\alpha-1}}$

$$\frac{w}{\alpha\tilde{n}^{\alpha-1}} = \frac{1-\beta\theta_p(1+\pi)^{\frac{\epsilon_p}{\alpha}}}{1-\beta\theta_p(1+\pi)^{\epsilon_p-1}}\left(\tilde{p}\frac{\epsilon_p-1}{\epsilon_p}\right) \equiv q \tag{61}$$

By (42)

$$f_1 = \frac{\frac{\epsilon_w-1}{\epsilon_w}\lambda\tilde{w}^{1-\epsilon_w}wh}{1-\beta\theta_w(1+\pi)^{\epsilon_w-1}}$$

By (45)

$$c = \left(\frac{h}{s}\right)^{\alpha}$$

By (34)

$$\lambda = c^{-\sigma} = \left(\frac{h}{s}\right)^{-\alpha\sigma}$$

Use this to eliminate λ from the expression for f_1

$$f_1 = \frac{\frac{\epsilon_w-1}{\epsilon_w}\left(\frac{h}{s}\right)^{-\alpha\sigma}\tilde{w}^{1-\epsilon_w}wh}{1-\beta\theta_w(1+\pi)^{\epsilon_w-1}}$$

By (43)

$$f_2 = \frac{\tilde{w}^{-\epsilon_w(1+\varphi)}h^{1+\varphi}}{1-\beta\theta_w(1+\pi)^{\epsilon_w(1+\varphi)}}$$

Use the expressions for f_1 and f_2 in (41) and solve for w

$$\begin{aligned} w &= \left(\frac{1-\beta\theta_w(1+\pi)^{\epsilon_w-1}}{1-\beta\theta_w(1+\pi)^{\epsilon_w(1+\varphi)}}\right)\left(\frac{\tilde{w}^{-\epsilon_w(1+\varphi)}}{\frac{\epsilon_w-1}{\epsilon_w}\tilde{w}^{1-\epsilon_w}s^{\sigma\alpha}}\right)h^{\sigma\alpha+\varphi} \\ &\equiv q_w h^{\sigma\alpha+\varphi} \end{aligned}$$

Now combine (38) and (45) to express $\tilde{n}$ in terms of h

$$\tilde{n} = \frac{\tilde{p}^{-\frac{\epsilon_p}{\alpha}}}{s}h \equiv q_{\tilde{n}}h$$

Finally, use these two expressions for w and $\tilde{n}$ in (61) and solve for the steady state value of h. This yields:

$$h = \left[\frac{\alpha q}{q_w q_{\tilde{n}}^{1-\alpha}}\right]^{\frac{1}{\sigma\alpha+\varphi+1-\alpha}}$$

1.13 Non-stochastic Steady State — with indexation

As in the case without indexation, we assign numerical values to the following 9 parameters θ_p, θ_w, σ, α, β, α_π, η_p, η_w, φ. In addition we give values to the indexation parameters χ^*, χ^p, ν^*, ν^p.

We also set, as before, the steady state values of π, z, μ, ξ.

Proceed as in the case without indexation. Solve (33) for i_t

$$(1+i) = \frac{(1+\pi)}{\beta}$$

By (40)

$$\pi^W = \pi$$

Let

$$A_w = (1+\pi^*)^{\nu^*+\nu^p}$$
$$A_p = (1+\pi^*)^{\chi^*+\chi^p}$$

By (50)

$$\tilde{p} = \left[\frac{1-\theta_p\left(\frac{A_p}{1+\pi}\right)^{1-\epsilon_p}}{1-\theta_p}\right]^{\frac{1}{1-\epsilon_p}}$$

By (53)

$$\tilde{w} = \left[\frac{1-\theta_w\left(\frac{A_w}{1+\pi^W}\right)^{1-\epsilon_w}}{1-\theta_w}\right]^{\frac{1}{1-\epsilon_w}}$$

By (54)

$$s = \left[\frac{(1-\theta_p)}{1-\theta_p\left(\frac{A_p}{1+\pi}\right)^{-\frac{\epsilon_p}{\alpha}}}\right]\tilde{p}^{\frac{-\epsilon_p}{\alpha}}$$

By (48)

$$x_1 = \frac{\tilde{p}^{1-\epsilon_p}}{1-\beta\theta_p\left(\frac{A_p}{1+\pi}\right)^{1-\epsilon_p}}c$$

By (49)

$$x_2 = \frac{\tilde{p}^{-\epsilon_p\frac{\epsilon_p}{\epsilon_p-1}}}{1-\beta\theta_p\left(\frac{A_p}{1+\pi}\right)^{-\frac{\epsilon_p}{\alpha}}}c\frac{w}{\alpha\tilde{n}^{\alpha-1}}$$

By (35)

$$x_1 = x_2$$

Solve the resulting expression for $q \equiv \frac{w}{\alpha \tilde{n}^{\alpha-1}}$

$$q \equiv \frac{w}{\alpha \tilde{n}^{\alpha-1}} = \frac{1 - \beta\theta_p \left(\frac{A_p}{1+\pi}\right)^{-\frac{\epsilon_p}{\alpha}}}{1 - \beta\theta_p \left(\frac{A_p}{1+\pi}\right)^{1-\epsilon_p}} \left(\frac{\epsilon_p - 1}{\epsilon_p}\right) \tilde{p} \tag{62}$$

By (51)

$$f_1 = \frac{\frac{\epsilon_w - 1}{\epsilon_w} \lambda \tilde{w}^{1-\epsilon_w} w h}{1 - \beta\theta_w \left(\frac{A_w}{1+\pi}\right)^{1-\epsilon_w}}$$

By (45)

$$c = \left(\frac{h}{s}\right)^{\alpha}$$

By (34)

$$\lambda = c^{-\sigma} = \left(\frac{h}{s}\right)^{-\alpha\sigma}$$

Use this to eliminate λ from the expression for f_1

$$f_1 = \frac{\frac{\epsilon_w - 1}{\epsilon_w} \left(\frac{h}{s}\right)^{-\alpha\sigma} \tilde{w}^{1-\epsilon_w} w h}{1 - \beta\theta_w \left(\frac{A_w}{1+\pi}\right)^{1-\epsilon_w}}$$

By (52)

$$f_2 = \frac{\tilde{w}^{-\epsilon_w(1+\varphi)} h^{1+\varphi}}{1 - \beta\theta_w \left(\frac{A_w}{1+\pi^W}\right)^{-\epsilon_w(1+\varphi)}}$$

Use the expressions for f_1 and f_2 in (41) and solve for w

$$\begin{aligned} w &= \left(\frac{1 - \beta\theta_w \left(\frac{A_w}{1+\pi}\right)^{1-\epsilon_w}}{1 - \beta\theta_w \left(\frac{A_w}{1+\pi}\right)^{-\epsilon_w(1+\varphi)}}\right) \left(\frac{\tilde{w}^{-\epsilon_w(1+\varphi)}}{\frac{\epsilon_w - 1}{\epsilon_w} \tilde{w}^{1-\epsilon_w} s^{\sigma\alpha}}\right) h^{\sigma\alpha+\varphi} \\ &\equiv q_w h^{\sigma\alpha+\varphi} \end{aligned}$$

Now combine (38) and (45) to express $\tilde{n}$ in terms of h

$$\tilde{n} = \frac{\tilde{p}^{-\frac{\epsilon_p}{\alpha}}}{s} h \equiv q_{\tilde{n}} h$$

Finally, use these two expressions for w and $\tilde{n}$ in (62) and solve for the steady state value of h. This yields:

$$h = \left[\frac{\alpha q}{q_w q_{\tilde{n}}^{1-\alpha}}\right]^{\frac{1}{\sigma\alpha+\varphi+1-\alpha}}$$