# Multiple Equilibria in Open Economies with Collateral Constraints 

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#### Abstract

This article establishes the existence of multiple equilibria in infinite-horizon open economy models in which the value of tradable and non-tradable endowments serves as collateral. In this environment, the economy displays self-fulfilling financial crises in which pessimistic views about the value of collateral induce agents to deleverage. Under plausible calibrations, there exist equilibria with underborrowing. This result stands in contrast to the overborrowing result stressed in the related literature. Underborrowing emerges in the present context because in economies that are prone to self-fulfilling financial crises, individual agents engage in excessive precautionary savings as a way to self-insure.


Key words: Pecuniary externalities, Collateral constraints, Multiple equilibria, Sudden stop, Overborrowing, Underborrowing, Self-fulfilling financial crises, Capital controls

JEL Codes: E44, F41, G01, H23

This article establishes the existence of multiple equilibria in open economies with flow collateral constraints in which the value of tradable and non-tradable income serves as collateral. It shows that these economies are prone to self-fulfilling financial crises that emerge as a result of pessimistic views about the value of collateral, which induce agents to deleverage. These crises have all the hallmarks of observed sudden stops: sharp depreciations of the real exchange rate, current account reversals, and contractions in domestic absorption. Further, they are more likely to occur during periods of weak economic fundamentals (depressed levels of output or unfavourable terms of trade).

In this class of models, multiple equilibria arise because, although the collateral constraint is well behaved at the individual level, in the sense that it tightens when individuals borrow more, it may be ill-behaved at the aggregate level, in the sense that it may relax as aggregate borrowing increases. The possibility that such a perverse relationship can give rise to multiple equilibria has

[^0]been suggested heuristically by Jeanne and Korinek (2010) in the context of an economy with a stock collateral constraint and by Mendoza (2005) in the context of an economy with a flow collateral constraint.

A result stressed in the related literature is that open economies with collateral constraints tend to overborrow, that is, they tend to borrow more than they would if the allocation were determined by a social planner who internalizes the effect of aggregate spending on the price of objects that serve as collateral (Auernheimer and García-Saltos, 2000; Jeanne and Korinek, 2010; Bianchi, 2011; Korinek, 2011, among others). This article shows that under empirically plausible calibrations the presence of multiple equilibria can give rise to underborrowing. Underborrowing can emerge because individual agents understand that they live in a fragile environment and as a result engage in excessive precautionary saving as a way to self-insure.

The second contribution of the article is quantitative. Existing quantitative studies avoid the multiplicity problem by choosing calibrations for which non-convexities are absent. This concern in choosing model parameterizations is explicitly mentioned, for instance, in Jeanne and Korinek (2010) in the context of a stock collateral constraint model and in Benigno et al. (2016) in the context of a flow collateral constraint model, and is implicit in the parameterizations adopted in Bianchi (2011) and Ottonello (2015), among others. The present article solves for equilibrium dynamics in the presence of non-convexities. We find that in the model economy studied by Bianchi, which is calibrated with parameter values typically used in the emerging-market business-cycle literature and fed with output shocks estimated on Argentine data, plausible variations of the parameter configuration give rise to equilibria in which the economy is prone to self-fulfilling crises.

A byproduct of this analysis is a diagnostic test that is readily applicable and can be of use to quantitative researchers seeking to ascertain whether their parameterizations give rise to multiplicity of equilibrium. This test is of particular use in avoiding parameter configurations that lead to non-convergence of standard algorithms for approximating equilibrium dynamics.

A third contribution of the article is to explicitly address the issue of implementation of the constrained optimal allocation. As is well known, policies that support the optimal allocation can also be consistent with other (non-optimal) allocations. A natural question is therefore what kind of policy can implement the optimal allocation. We provide two examples, one involving ex post intervention and one involving ex ante interventions. The former consists of a capital control tax feedback rule whereby the tax on external debt is an increasing function of the change in the net external debt position. The latter consists of a constant proportional subsidy to consumption of non-tradable goods. Both of these policy schemes can avoid self-fulfilling financial crises and implement the optimal allocation.

This article is related to several branches of the literature on credit frictions in macroeconomics. The type of flow collateral constraint we study was introduced in open economy models by Mendoza (2002) to understand sudden stops caused by fundamental shocks. The externality that emerges when debt is denominated in tradables goods but partly leveraged on non-tradable income and the consequent room for macroprudential policy was emphasized by Korinek (2007) in the context of a three-period model. Bianchi (2011) shows, in the context of an infinite-horizon model, that the externality can lead to overborrowing and characterizes the behaviour of optimal prudential policy. An exception to the standard overborrowing result is Benigno et al. (2013). However, the cause of underborrowing in the Benigno et al. model is of a different nature from the one identified in the present paper. In their work, underborrowing stems from introducing production in the non-tradable sector or distortionary taxation. The result of the Benigno et al. paper is complementary but different from the one presented here. In the present study,

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underborrowing arises even in the context of an endowment economy and without distortionary taxation and is due to the fragility created by the possibility of self-fulfilling crises.

The remainder of the article is organized as follows. Section 2 presents an open economy with a flow collateral constraint in which tradable and non-tradable output have collateral value. Section 3 characterizes steady-state equilibria. Section 4 characterizes analytically multiplicity of equilibrium. It shows the existence of up to two equilibria with self-fulfilling crashes in the value of collateral. Section 5 quantitatively characterizes the dynamics in a stochastic economy with output shocks. It establishes that self-fulfilling deleveraging crises occur under plausible calibrations and resemble observed sudden stops. It also shows that equilibrium multiplicity can give rise to underborrowing in equilibrium. Section 6 introduces non-fundamental uncertainty (sunspots) and shows that it can generate persistent self-fulfilling financial crises. Section 7 studies policies that implement the optimal allocation. Section 8 concludes.

## 2. THE MODEL

Consider a small open endowment economy in which households have preferences of the form

$$
\begin{equation*}
\mathbb{E}_{0} \sum_{t=0}^{\infty} \beta^{t} U\left(c_{t}\right) \tag{1}
\end{equation*}
$$

where $c_{t}$ denotes consumption in period $t, U(\cdot)$ denotes an increasing and concave period utility function, $\beta \in(0,1)$ denotes the subjective discount factor, and $\mathbb{E}_{t}$ denotes the expectations operator conditional on information available in period $t$. The period utility function takes the CRRA form $U(c)=\left(c^{1-\sigma}-1\right) /(1-\sigma)$ with $\sigma>0$. We assume that consumption is a composite of tradable and non-tradable goods, taking the CES form

$$
\begin{equation*}
c_{t}=A\left(c_{t}^{T}, c_{t}^{N}\right) \equiv\left[a c_{t}^{T^{1-1 / \xi}}+(1-a) c_{t}^{N^{1-1 / \xi}}\right]^{1 /(1-1 / \xi)} \tag{2}
\end{equation*}
$$

with $\xi>0, a \in(0,1)$, and where $c_{t}^{T}$ denotes consumption of tradables in period $t$ and $c_{t}^{N}$ denotes consumption of non-tradables in period $t$. Households are assumed to have access to a single, one-period, risk-free, internationally traded bond denominated in terms of tradable goods that pays the interest rate $r$ when held from period $t$ to period $t+1$. The household's sequential budget constraint is given by

$$
\begin{equation*}
c_{t}^{T}+p_{t} c_{t}^{N}+d_{t}=y_{t}^{T}+p_{t} y_{t}^{N}+\frac{d_{t+1}}{1+r} \tag{3}
\end{equation*}
$$

where $d_{t}$ denotes the amount of debt assumed in period $t-1$ and due in period $t, p_{t}$ denotes the relative price of non-tradables in terms of tradables, and $y_{t}^{T}$ and $y_{t}^{N}$ denote the endowments of tradables and non-tradables, respectively. Both endowments are assumed to be exogenously given. Movements in $y_{t}^{T}$ can be interpreted as disturbances to the country's terms of trade.

The collateral constraint takes the form

$$
\begin{equation*}
d_{t+1} \leq \kappa\left(y_{t}^{T}+p_{t} y_{t}^{N}\right), \tag{4}
\end{equation*}
$$

where $\kappa>0$ is a parameter. Throughout this paper, we will assume that $\kappa<(1+r) / r$, where $r$ is the steady-state real interest rate. This assumption makes the collateral constraint economically relevant, in the sense that values of $\kappa$ higher than $(1+r) / r$ would imply that the collateral constraint is slack even at the natural debt limit.

The borrowing constraint introduces a pecuniary externality, because each individual household takes the real exchange rate, $p_{t}$, as exogenously determined, even though, collectively, their absorptions of non-tradable goods are a key determinant of this relative price. From the perspective of the individual household, the collateral constraint is well behaved in the sense that the higher the debt level is, the tighter the collateral constraint will be. As we shall see shortly, however, this may not be the case in equilibrium.

Households choose processes $c_{t}^{T}>0, c_{t}^{N}>0, c_{t}>0$, and $d_{t+1}$ to maximize (1) subject to (2)(4), given the processes $p_{t}, y_{t}^{T}$, and $y_{t}^{N}$ and the initial debt position $d_{0}$. The first-order conditions of this problem are (2)-(4) and

$$
\begin{gather*}
U^{\prime}\left(A\left(c_{t}^{T}, c_{t}^{N}\right)\right) A_{1}\left(c_{t}^{T}, c_{t}^{N}\right)=\lambda_{t}  \tag{5}\\
p_{t}=\frac{1-a}{a}\left(\frac{c_{t}^{T}}{c_{t}^{N}}\right)^{1 / \xi}  \tag{6}\\
\left(\frac{1}{1+r}-\mu_{t}\right) \lambda_{t}=\beta \mathbb{E}_{t} \lambda_{t+1}  \tag{7}\\
\mu_{t} \geq 0 \tag{8}
\end{gather*}
$$

and

$$
\begin{equation*}
\mu_{t}\left[d_{t+1}-\kappa\left(y_{t}^{T}+p_{t} y_{t}^{N}\right)\right]=0 \tag{9}
\end{equation*}
$$

where $\beta^{t} \lambda_{t}$ and $\beta^{t} \lambda_{t} \mu_{t}$ denote the Lagrange multipliers on the sequential budget constraint (3) and the collateral constraint (4), respectively. As usual, the Euler equation (7) equates the marginal benefit of assuming more debt with its marginal cost. During tranquil times, when the collateral constraint does not bind, one unit of debt payable in $t+1$ increases tradable consumption by $1 /(1+r)$ units in period $t$, which increases utility by $\lambda_{t} /(1+r)$. At the optimal intertemporal allocation, this marginal benefit must be equal to the expected marginal cost of debt assumed in period $t$ and payable in $t+1$, which is given by the expected marginal utility of consumption in period $t+1$ discounted at the subjective discount factor, $\beta \mathbb{E}_{t} \lambda_{t+1}$. During financial crises, when the collateral constraint binds, the marginal utility of debt falls from $\lambda_{t} /(1+r)$ to $[1 /(1+r)-$ $\left.\mu_{t}\right] \lambda_{t}$, reflecting a shadow penalty for trying to increase debt when the collateral constraint is binding.

In equilibrium, the market for non-tradables must clear. That is,

$$
c_{t}^{N}=y_{t}^{N} .
$$

Using this expression and equations (5) and (6) to eliminate $c_{t}^{N}, \lambda_{t}$, and $p_{t}$ from the household's first-order conditions, we can define a competitive equilibrium as a set of processes $\left\{c_{t}^{T}, d_{t+1}, \mu_{t}\right\}$ satisfying

$$
\begin{gather*}
\left(\frac{1}{1+r}-\mu_{t}\right) U^{\prime}\left(A\left(c_{t}^{T}, y_{t}^{N}\right)\right) A_{1}\left(c_{t}^{T}, y_{t}^{N}\right)=\beta \mathbb{E}_{t} U^{\prime}\left(A\left(c_{t+1}^{T}, y_{t+1}^{N}\right)\right) A_{1}\left(c_{t+1}^{T}, y_{t+1}^{N}\right),  \tag{10}\\
c_{t}^{T}+d_{t}=y_{t}^{T}+\frac{d_{t+1}}{1+r}  \tag{11}\\
d_{t+1} \leq \kappa\left[y_{t}^{T}+\left(\frac{1-a}{a}\right) c_{t}^{T^{1 / \xi}} y_{t}^{N^{1-1 / \xi}}\right] \tag{12}
\end{gather*}
$$

$$
\begin{gather*}
\mu_{t}\left[\kappa y_{t}^{T}+\kappa\left(\frac{1-a}{a}\right) c_{t}^{\left.T^{1 / \xi} y_{t}^{N^{1-1 / \xi}}-d_{t+1}\right]=0,}\right.  \tag{13}\\
\mu_{t} \geq 0 \tag{14}
\end{gather*}
$$

and

$$
\begin{equation*}
c_{t}^{T}>0 \tag{15}
\end{equation*}
$$

given the exogenous processes $\left\{y_{t}^{T}, y_{t}^{N}\right\}$ and the initial condition $d_{0}$.
The fact that $c_{t}^{T}$ appears on the right-hand side of the equilibrium version of the collateral constraint, equilibrium condition (12), means that when the absorption of tradables falls the collateral constraint endogenously tightens. Individual agents do not take this effect into account in choosing their consumption plans. This is the nature of the pecuniary externality in this model.

As we saw earlier, from the individual agent's perspective the collateral constraint is well behaved in the sense that it tightens as the level of debt increases. This may not be the case at the aggregate level. To see this, use equilibrium condition (11) to eliminate $c_{t}^{T}$ from equilibrium condition (12) to obtain

$$
d_{t+1} \leq \kappa\left[y_{t}^{T}+\left(\frac{1-a}{a}\right)\left(y_{t}^{T}+\frac{d_{t+1}}{1+r}-d_{t}\right)^{1 / \xi} y_{t}^{N}{ }^{1-1 / \xi}\right] .
$$

It is clear from this expression that the right-hand side is increasing in the equilibrium level of external debt, $d_{t+1}$. Moreover, depending on the values assumed by the parameters $\kappa$, $a$, and $\xi$, the right-hand side may increase more than one for one with $d_{t+1}$. In this case an increase in debt, instead of tightening the collateral constraint may relax it. In other words, the more indebted the economy becomes, the less leveraged it will be. As we will see shortly, this possibility can give rise to multiple equilibria and self-fulfilling drops in the value of collateral.

Furthermore, while the individual household's constraints represent a convex set, the equilibrium aggregate resource constraint may not. To see this, examine first the restrictions faced by the individual household. If two debt levels $d^{1}$ and $d^{2}$ satisfy (3) and (4), then any weighted average $\alpha d^{1}+(1-\alpha) d^{2}$ for $\alpha \in[0,1]$ also satisfies these two conditions. From an equilibrium perspective, however, this ceases to be true in general. If the intratemporal elasticity of substitution $\xi$ is less than unity, which is the case of greatest empirical relevance for many countries (Akinci, 2011), the equilibrium value of collateral is convex in the level of debt. This property may cause the emergence of two distinct values of $d_{t+1}$ for which the collateral constraint binds and two disjoint intervals of debt levels for which the collateral constraint is slack, rendering the feasible set of debts non-convex.

To analytically characterize conditions for the existence of self-fulfilling financial crises, we impose the following assumptions, which will be relaxed in the quantitative analysis: the tradable and non-tradable endowments are constant and equal to $y_{t}^{T}=y^{T}$ and $y_{t}^{N}=1$, for all $t$, respectively. Finally, we set $\beta(1+r)=1$. Given these assumptions, the equilibrium conditions (10)-(15) can be written as

$$
\begin{gather*}
\Lambda\left(c_{t}^{T}\right)\left[1-(1+r) \mu_{t}\right]=\Lambda\left(c_{t+1}^{T}\right),  \tag{16}\\
c_{t}^{T}+d_{t}=y^{T}+\frac{d_{t+1}}{1+r}  \tag{17}\\
d_{t+1} \leq \kappa\left[y^{T}+\frac{(1-a)}{a}\left(y^{T}+\frac{d_{t+1}}{(1+r)}-d_{t}\right)^{\frac{1}{\xi}}\right], \tag{18}
\end{gather*}
$$

$$
\begin{gather*}
\mu_{t}\left\{\kappa\left[y^{T}+\frac{(1-a)}{a}\left(y^{T}+\frac{d_{t+1}}{(1+r)}-d_{t}\right)^{\frac{1}{\xi}}\right]-d_{t+1}\right\}=0  \tag{19}\\
\mu_{t} \geq 0 \tag{20}
\end{gather*}
$$

and

$$
\begin{equation*}
c_{t}^{T}>0 \tag{21}
\end{equation*}
$$

with $d_{0}$ given, where

$$
\Lambda\left(c_{t}^{T}\right) \equiv U^{\prime}\left(A\left(c_{t}^{T}, 1\right)\right) A_{1}\left(c_{t}^{T}, 1\right)
$$

denotes the equilibrium level of the marginal utility of tradable consumption. Given the assumed concavity of $U(\cdot)$ and $A(\cdot, \cdot), \Lambda(\cdot)$ is a decreasing function.

## 3. STEADY-STATE EQUILIBRIA

Here, we characterize conditions under which, given $d_{0}$, there exists an equilibrium in which traded consumption and debt are constant for all $t \geq 0$, that is, an equilibrium in which $c_{t}^{T}=c_{0}^{T}$ and $d_{t}=d_{0}$ for all $t \geq 0$, where $d_{0}$ is a given initial condition. We refer to this equilibrium as a steady-state equilibrium. ${ }^{1}$ By (16), we have that in a steady-state equilibrium $\mu_{t}=0$ for all $t$. This means that in a steady-state equilibrium the slackness condition (19) and the non-negativity constraint (20) are also satisfied for all $t$. When $d_{t+1}=d_{t}=d$, the collateral constraint (18) becomes

$$
\begin{equation*}
d \leq \kappa\left[y^{T}+\frac{(1-a)}{a}\left(y^{T}-\frac{r}{(1+r)} d\right)^{\frac{1}{\xi}}\right] . \tag{22}
\end{equation*}
$$

We refer to this expression as the steady-state collateral constraint. Figure 1 displays the left- and right-hand sides of the steady-state collateral constraint as a function of $d$. The left-hand side is the 45 -degree line. The right-hand side, shown with a thick solid line, is the steady-state value of collateral. By (17), steady-state consumption of tradables is given by $c^{T}=y^{T}-\frac{r}{1+r} d$. Define the natural debt limit, denoted $\bar{d}$, as the value of $d$ associated with $c^{T}=0$, that is, $\bar{d} \equiv y^{T}(1+r) / r$. By equilibrium condition (21), $c^{T}$ must be positive. This means that a steady-state equilibrium can only exist for $d<\bar{d}$. For values of debt between zero and $\bar{d}$ the right-hand side of (22) is downward sloping (recall that $\xi>0$ ). It follows that the steady-state collateral constraint is well behaved in the sense that the higher the steady-state level of debt is, the tighter the steady-state collateral constraint will be. The left- and right-hand sides of (22) intersect once somewhere in the interval $(0, \bar{d})$. To see this, note first that the left-hand side of the steady-state collateral constraint is upward sloping while the right-hand side is downward sloping. Second, at $d=\bar{d}$, the left-hand side of (22) is larger than the right-hand side, and at $d=0$ the left-hand side is smaller than the right-hand side. ${ }^{2}$

1. Note that steady-state equilibrium and steady state are not the same concepts. To illustrate the difference, consider an economy whose equilibrium dynamics are described by the expression $H\left(x_{t+1}, x_{t}\right)=0$, where $x_{t}$ is an endogenous state variable. Then, the economy has a steady state if there is a constant $x$ such that $H(x, x)=0$. Given an initial condition $x_{0}$, the economy has a steady-state equilibrium if $H\left(x_{0}, x_{0}\right)=0$. Thus, for example, given $x_{0}$, the economy may have a steady state, but may or may not have a steady-state equilibrium.
2. Specifically, at $d=\bar{d}$, the right-hand side of the steady-state collateral constraint equals $\kappa y^{T}$ and the left-hand side equals $(1+r) / r y^{T}>\kappa y^{T}$, where the inequality follows from the maintained assumption that $\kappa<(1+r) / r$. At $d=0$, the left-hand side of the steady-state collateral constraint is 0 and the right-hand side is equal to $\kappa y^{T}+\kappa(1-a) / a y^{T / \xi}>0$.


Figure 1
Feasible debt levels in the steady state
Notes: The 45-degree line is the left-hand side of the steady-state collateral constraint, equation (22), and the thick line is its right-hand side. The value $\bar{d}$ represents the natural debt limit, and $\tilde{d}$ is the maximum level of debt consistent with a steady-state equilibrium.

Let $\tilde{d}<\bar{d}$ be the value of $d$ at which the steady-state collateral constraint (22) holds with equality, point $X$ in Figure 1. Formally, $\tilde{d}$ is implicitly given by

$$
\begin{equation*}
\tilde{d}=\kappa\left[y^{T}+\frac{1-a}{a}\left(y^{T}-\frac{r}{1+r} \tilde{d}\right)^{\frac{1}{\xi}}\right] \tag{23}
\end{equation*}
$$

Any value of initial debt, $d_{0}$, less than or equal to $\tilde{d}$ satisfies the steady-state collateral constraint (22). Since we have already shown that a constant value of debt less than $\bar{d}$ also satisfies all other equilibrium conditions, we have demonstrated that any initial value of debt less than or equal to $\tilde{d}$ can be supported as a steady-state equilibrium.

## 4. SELF-FULFILLING FINANCIAL CRISES

Do there exist equilibria other than the steady-state equilibrium? The answer turns out to be yes. To show this, we characterize conditions under which a second equilibrium exists with the property that the collateral constraint binds in period 0 . In this equilibrium, for non-fundamental reasons, in period 0 agents wake up feeling pessimistic about the economy and decide to cut consumption, increase savings, and deleverage. In turn, the contraction in consumption brings down the relative price of non-tradables, causing the value of collateral to fall and the collateral constraint to bind, validating agents' pessimistic sentiments. Because of these characteristics, we refer to this second equilibrium as a self-fulfilling financial-crisis equilibrium:


Figure 2
Multiple equilibria: self-fulfilling financial crisis
Notes: The downward sloping solid line is the right-hand side of the steady-state collateral constraint (22). The upward-sloping broken line is the right-hand side of the period- 0 collateral constraint (24). These two lines intersect at point A, which represents a steady-state equilibrium. At point $B$, the period- 0 collateral constraint is binding and the economy experiences a self-fulfilling financial crisis.

Definition 1 (Self-Fulfilling Financial-Crisis Equilibrium) For any initial level of debt $d_{0}<$ $\tilde{d}$, a self-fulfilling financial-crisis equilibrium is a set of deterministic paths $\left\{c_{t}^{T}, d_{t+1}, \mu_{t}\right\}_{t=0}^{\infty}$ satisfying conditions (16)-(21) and $d_{1}<d_{0}$ (deleveraging), where $\tilde{d}$ is defined in equation (23).

To facilitate the analysis, we focus on self-fulfilling crises that take place and end in period $0 .{ }^{3}$ Thus, in period 1 the economy reaches a steady state ( $d_{t}=d_{1}$, for $t \geq 2$ ). Consider the collateral constraint, equation (18), in period 0 , which is given by

$$
\begin{equation*}
d_{1} \leq \kappa\left[y^{T}+\left(\frac{1-a}{a}\right)\left(y^{T}+\frac{d_{1}}{1+r}-d_{0}\right)^{\frac{1}{\xi}}\right] \tag{24}
\end{equation*}
$$

Suppose that $d_{0}<\tilde{d}$, so that a steady-state equilibrium exists. The right-hand side of the period-0 collateral constraint is increasing in $d_{1}$, which means that the value of collateral is increasing in debt. This is so because more borrowing in period 0 (i.e. a larger value of $d_{1}$ ) allows for higher consumption, which in equilibrium leads to an increase in the relative price of non-tradables.

Figure 2 plots with a broken line the right-hand side of the period- 0 collateral constraint as a function of $d_{1}$. The figure assumes that $0<\xi<1$, which implies that the right-hand side of
3. In Section 6, we show that this is the only type of self-fulfilling crisis that can occur under perfect foresight. There we also show that there exist sunspot equilibria, in which crises last for more than one period.
the period- 0 collateral constraint is a convex function of $d_{1}$. The figure also reproduces from Figure 1, with a thick solid line, the right-hand side of the steady-state collateral constraint. The right-hand sides of the period-0 and steady-state collateral constraints intersect when $d_{1}=d_{0}$ (point A in the figure). In fact, point $A$ is the steady-state equilibrium. If the economy stays forever at point $A$, the collateral constraint is always slack, and debt is constant and equal to $d_{0}$ at all times.

We now show that point $B$ in Figure 2 is a self-fulfilling financial-crisis equilibrium. In this equilibrium, for no fundamental reason, the value of collateral falls in period 0 and the collateral constraint binds, forcing agents to deleverage from $d_{0}$ to $d_{1}^{b}<d_{0}$. Debt then stays at this lower level forever, as the economy reaches a steady state in period 1 . The deleveraging in period 0 requires a cut in consumption and an increase in the current account, so the dynamics have all the marks of a sudden stop.

To establish this result, we must show that the proposed path of debt induces paths of $c_{t}^{T}$ and $\mu_{t}$ that satisfy equilibrium conditions (16)-(21). Suppose for now that at point $B c_{0}^{T}$ is positive, so that equilibrium condition (21) is satisfied. Because at point $B$ the period- 0 collateral constraint is binding, equilibrium conditions (18) and (19) are satisfied. Further, the fact that in the proposed equilibrium $d_{1}<d_{0}$ and $d_{1}=d_{2}$ implies that $c_{0}^{T}<c_{1}^{T}$, which can be verified by comparing the resource constraint (17) evaluated at $t=0$ and $t=1$. In turn, $c_{0}^{T}<c_{1}^{T}$ implies, by the Euler equation (16), that the Lagrange multiplier $\mu_{0}$ is strictly positive. As a result, the nonnegativity constraint (20) holds in period 0 . This establishes that if the debt level associated with point $B$ implies positive consumption in period 0 , then it satisfies all equilibrium conditions in period 0 . To establish that all equilibrium conditions are also satisfied from period 1 onward, note that since $d_{1}^{b}<\tilde{d}$, the analysis of steady-state equilibria in Section 3 guarantees that $d_{t}=d_{1}^{b}$ for all $t \geq 1$ represents a steady-state equilibrium.

Thus far, we have assumed that point $B$ in Figure 2 exists and that consumption of tradables in period 0 is positive. That is, we have assumed that there exists an initial level of debt $d_{0}$ such that the right-hand side of the period- 0 collateral constraint crosses the 45 -degree line to the left of $d_{0}$ and that at the crossing $c_{0}^{T}$ is positive. We now provide conditions under which this is indeed the case. To this end, consider the initial level of debt $d_{0}=\tilde{d}$. Figure 3 plots with a broken line the right-hand side of the period-0 collateral constraint associated with $d_{0}=\tilde{d}$ as a function of $d_{1}$, again assuming that $0<\xi<1$. This function is upward sloping and convex and crosses both the 45 -degree line and the right-hand side of the steady-state collateral constraint (thick solid line) at point $X$, where $d_{1}=d_{0}=\tilde{d}$. From the analysis of steady-state equilibria of Section 3, we have that $d_{t}=d_{0}=\tilde{d}$ for all $t \geq 0$ is an equilibrium. In particular, equilibrium condition (21), which requires that consumption be positive, is satisfied because $\tilde{d}$ is lower than the natural debt limit $\bar{d}$. Suppose that, as assumed in the figure, the slope of the right-hand side of the period0 collateral constraint is greater than 1 at $d_{1}=\tilde{d}$ when $d_{0}=\tilde{d}$. Now consider an initial value of debt $d_{0}=d^{\prime}$ slightly lower than $\tilde{d}$. The right-hand side of the period- 0 collateral constraint associated with $d_{0}=d^{\prime}$ is located to the left of the one associated with $d_{0}=\tilde{d}$, as shown with a dash-dotted locus in Figure 3. By continuity, if $\tilde{d}-d^{\prime}$ is sufficiently small, the right-hand side of the period-0 collateral constraint associated with $d_{0}=d^{\prime}$ will also cross the 45 -degree line with a slope greater than one, point $B$ in the figure. Also, point $B$ will be located left of point $A$. Furthermore, since consumption of tradables in period 0 is strictly positive at point $X$, continuity ensures that it is also positive at point $B$. Thus, point $B$ represents a self-fulfilling financial-crisis equilibrium.

Taking stock, we have that if the slope of the right-hand side of the period- 0 collateral constraint evaluated at $d_{1}=d_{0}=\tilde{d}$ is greater than 1 , then, given $0<\xi<1$, for initial debt levels $d_{0}$ in a neighbourhood left of $\tilde{d}$, the steady-state equilibrium coexists with a self-fulfilling financial-crisis


Figure 3
Existence of multiple equilibria
Notes: The downward-sloping solid line is the right-hand side of the steady-state collateral constraint, given in equation (22). The upwardsloping broken and dashed-dotted lines are the right-hand sides of the period-0 collateral constraint, given in equation (24) for $d_{0}=\tilde{d}$ and $d_{0}=d^{\prime}<\tilde{d}$, respectively. The figure is drawn under the assumptions $0<\xi<1, S(\tilde{d} ; \tilde{d})>1$, and $\left|d^{\prime}-\tilde{d}\right|$ sufficiently small.
equilibrium. To establish this result more formally, consider the value of period- 1 debt, $d_{1}^{b}$, at which the period- 0 collateral constraint binds, which is implicitly given by

$$
d_{1}^{b}=\kappa\left[y^{T}+\frac{(1-a)}{a}\left(y^{T}+\frac{d_{1}^{b}}{(1+r)}-d_{0}\right)^{\frac{1}{\xi}}\right]
$$

By (23), if $d_{0}=\tilde{d}$, then $d_{1}^{b}=\tilde{d}$. In order for $d_{1}^{b}$ to fall by more than $d_{0}$ as $d_{0}$ falls slightly below $\tilde{d}$, we need that the derivative of $d_{1}^{b}$ with respect to $d_{0}$ evaluated at $d_{0}=\tilde{d}$ be larger than one, that is,

$$
\begin{equation*}
\left.\frac{d d_{1}^{b}}{d d_{0}}\right|_{d_{0}=\tilde{d}}=\frac{(1+r) S(\tilde{d} ; \tilde{d})}{S(\tilde{d} ; \tilde{d})-1}>1, \tag{25}
\end{equation*}
$$

where

$$
\begin{equation*}
S\left(d_{1} ; d_{0}\right) \equiv \kappa\left(\frac{1-a}{a}\right) \frac{1}{(1+r)} \frac{1}{\xi}\left(y^{T}+\frac{d_{1}}{1+r}-d_{0}\right)^{\frac{1}{\xi}-1} \tag{26}
\end{equation*}
$$

denotes the slope of the right-hand side of the period-0 collateral constraint as a function of $d_{1}$ for a given value of $d_{0}$. Noting that $y^{T}+\frac{d_{1}}{1+r}-d_{0}=c_{0}^{T}$, we have that for any level of $d_{1}$ consistent with positive consumption, $S\left(d_{1} ; d_{0}\right)$ is positive and increasing in $d_{1} .{ }^{4}$ Thus, $S(\tilde{d} ; \tilde{d})>0$.
4. Recall the maintained assumption that $0<\xi<1$.

Then, from (25) we have that $\left.\frac{d d_{1}^{b}}{d d_{0}}\right|_{d_{0}=\tilde{d}}>1$ if and only if $S(\tilde{d} ; \tilde{d})>1$. This establishes that $S(\tilde{d} ; \tilde{d})>$ 1 is a sufficient condition for the existence of a self-fulfilling financial-crisis equilibrium. It does not show necessity, because of the local nature of the argument. In Appendix A, we show that when $\xi \in(0,1)$, the condition $S(\tilde{d} ; \tilde{d})>1$ is indeed necessary and sufficient. Furthermore, there we characterize an interval containing all the initial values of debt associated with self-fulfilling financial-crisis equilibria. We summarize this result in the following proposition:

Proposition 1 (Existence of Self-Fulfilling Financial-Crisis Equilibria) Suppose $y_{t}^{N}=1$, $y_{t}^{T}=y^{T}>0, \beta(1+r)=1$, and $\xi \in(0,1)$. Then, the steady-state equilibrium coexists with a selffulfilling financial-crisis equilibrium if and only if $S(\tilde{d} ; \tilde{d})>1$ and $d_{0} \in\left[\hat{d}_{0}, \tilde{d}\right)$, where $S(\cdot ; \cdot)$ is the slope of the right-hand side of the period-0 collateral constraint given in equation (26) and $\tilde{d}$ and $\hat{d}_{0}$ are given in equations (23) and (A.3), respectively.

Proof: See Section A of the Appendix.
Multiplicity of equilibrium and the existence of self-fulfilling financial crises is not limited to the case of an intratemporal elasticity of substitution less than unity, $0<\xi<1$. In Section C of the Appendix, we show that steady-state equilibria can also co-exist with self-fulfilling financial crisis equilibria in the case that $\xi \geq 1$.

We now show that in the present economy, self-fulfilling financial crises come in two sizes. Specifically, for a given initial level of debt, $d_{0}$, there may be more than one self-fulfilling financialcrisis equilibrium. The right-hand side of the period-0 collateral constraint might cross the 45degree line twice with a positive slope, as shown in Figure 4. From equation (26), we have that if the second crossing (point $C$ in the figure) occurs at a level of $d_{1}$ higher than the one at which the slope of the right-hand side of the period- 0 collateral constraint is zero, consumption of tradables in period $0, c_{0}^{T}$, is positive at both crossings (points $B$ and $C$ ). In this case, both crossings represent self-fulfilling financial-crisis equilibria. In turn, these two equilibria coexist with the steady-state equilibrium (point $A$ ).

Both self-fulfilling-crisis equilibria feature deleveraging ( $d_{1}<d_{0}$ ) and a binding collateral constraint in period 0 . The equilibrium associated with point $C$ entails a larger drop in the value of collateral and more deleveraging in period 0 than the equilibrium associated with point $B$. Thus, the intensity of people's concerns about the economic outlook can give rise to crises of different magnitude. Corollary 1 provides necessary and sufficient conditions for the existence of two self-fulfilling financial-crisis equilibria. It also provides the range of initial debt levels, $d_{0}$, for which multiple self-fulfilling financial-crisis equilibria exist.

Corollary 1 (Existence of Two Self-Fulfilling Financial-Crisis Equilibria) Two self-fulfilling financial-crisis equilibria exist if and only if the conditions of Proposition 1 are satisfied. The range of initial debt levels, $d_{0}$, for which two self-fulfilling financial-crisis equilibria exist is $\left[\hat{d}_{0}, \min \left(\left(1+\frac{\kappa}{1+r}\right) y^{T}, \tilde{d}\right)\right)$, where $\hat{d}_{0}$ is defined in Proposition 1, and $\tilde{d}$ is defined in equation (23).

Proof: See Appendix B.
In words, Corollary 1 says that if there exist initial debt levels for which one self-fulfilling financial-crisis equilibrium exists, then there also exist initial debt levels for which two such equilibria exist.


Figure 4
Multiple self-fulfilling financial-crisis equilibria
Notes: The downward-sloping solid line is the right-hand side of the steady-state collateral constraint, given in equation (22). The upwardsloping broken line is the right-hand side of the period- 0 collateral constraint, given in equation (24). The figure is drawn for the case that $0<\xi<1$.

The intuition behind the existence of self-fulfilling financial-crisis equilibria is as follows. Imagine the economy being originally in a steady state with debt constant and equal to $d_{0}$. Unexpectedly, the public becomes pessimistic and aggregate demand contracts. The contraction in aggregate demand means that households want to consume less of both types of good, tradable and non-tradable. Tradables can always be sold abroad, but non-tradables must be sold exclusively in the domestic market. Thus, the fall in the demand for consumption goods causes a decline in the relative price of non-tradables, $p_{0}$. As a result, the value of collateral, given by $\kappa\left(y^{T}+p_{0} y^{N}\right)$, also falls. The reduction in collateral is so large that it forces households to deleverage. To reduce their net debt positions, households must cut spending, validating the initial pessimistic sentiments, and making the financial crisis self-fulfilling. The contraction in the debt position and the fall in the relative price of non-tradables imply that the self-fulfilling financial crisis occurs in the context of a current account improvement and a depreciation of the real exchange rate.

It is worth pointing out that the possibility of self-fulfilling crises is not unrelated to fundamentals. Weak fundamentals can open the door to non-fundamental crises. To see this, note that the right-hand side of the period-0 collateral constraint shifts up in response to an increase in the endowment of tradables, $y^{T}$. Thus, given an initial debt level $d_{0}$, for sufficiently high values of $y^{T}$, the period-0 collateral constraint could be slack for all levels of $d_{1}$, but for lower values of $y^{T}$ it could have one or two crossings of the type B or C discussed above.

Although the model studied in this section is stylized, it is of interest to see whether selffulfilling financial crises exist for reasonable parameterizations. To facilitate comparison with the related literature, we adopt as the starting point the calibration of Bianchi (2011). Accordingly, we

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Figure 5
Existence of multiple equilibria for different parameterizations of the model
Notes: $X$ baseline parameterization; $Y$ value at which $S(\tilde{d} ; \tilde{d})$ takes the value 1 . The model displays multiple equilibria if $S(\tilde{d} ; \tilde{d})>1$. In each panel, all parameters other than the one displayed on the horizontal axis are fixed at their baseline values $(\xi=0.83, \kappa=0.33, a=0.31$, $r=0.04$, and $y^{T}=y^{N}=1$ ).
set the time unit to one year, the interest rate to $4 \%(r=0.04)$, the leverage parameter $\kappa$ to $0.33,{ }^{5}$ the intratemporal elasticity of substitution, $\xi$, to 0.83 , the share parameter $a$ to 0.31 , and $y^{T}=y^{N}=1$. With these values in hand, one can calculate the slope $S(\tilde{d} ; \tilde{d})$ by first using equation (23) to find $\tilde{d}$ and then evaluating equation (26) at $d_{0}=d_{1}=\tilde{d}$. This yields $S(\tilde{d} ; \tilde{d})=0.85$, which implies that the conditions for multiplicity of equilibria are not satisfied under this calibration.

Figure 5 explores the existence of self-fulfilling financial-crisis equilibria around this calibration. Each panel displays the value of $S(\tilde{d} ; \tilde{d})$ as a function of a particular parameter, holding all other parameters at their baseline values. The top-left panel of the figure shows that the less substitutable tradables and non-tradables are, the more likely it is that self-fulfilling financial-crisis equilibria exist. The smaller is the intratemporal elasticity of substitution $\xi$, the larger will be the decline in the relative price of non-tradables, $p$, required to clear the market in response to a decrease in desired absorption. In turn, because $p$ determines the value of collateral, we have that the smaller $\xi$ is, the steeper the slope of the collateral constraint will be. The figure shows that multiple equilibria exist for values of $\xi$ below 0.7 . This value lies within the range of values that are empirically relevant. For example, Bianchi (2011) reports that estimates of $\xi$ range from 0.4 to 0.83 and justifies his calibration of $\xi$ equal to 0.83 with the desire to minimize
5. Bianchi's formulation of the collateral constraint is $d_{t+1} /(1+r)=0.32\left(y_{t}^{T}+p_{t} y_{t}^{N}\right)$, which corresponds to setting $\kappa=0.32(1+r)$ in our formulation.


Figure 6
Debt policy functions
Notes: Each panel displays the equilibrium value of $d_{1}$ as a function of $d_{0}$. The figure is drawn using the parameter values $\kappa=0.33$, $a=0.31, r=0.04$, and $y^{T}=y^{N}=1$. When $\xi=0.5, S(\tilde{d}, \tilde{d})>1$, and when $\xi=0.83, S(\tilde{d}, \tilde{d})<1$.
the size of the pecuniary externality. Similarly, Akinci (2011) surveys the empirical evidence on estimates of $\xi$ for emerging countries and finds that the typical value is around 0.5 .

The top-right panel of the figure shows that the slope $S(\tilde{d} ; \tilde{d})$ is increasing in $\kappa$, which says that the emergence of multiple equilibria is more likely the higher $\kappa$ is. This result is intuitive, because $\kappa$ represents the fraction of income that is pledgeable as collateral. Thus, $\kappa$ captures the sensitivity of collateral with respect to income. Self-fulfilling crises emerge for values of $\kappa$ above 0.39 , which is a debt ratio well within the range of values observed in emerging countries. The lower-left panel shows that multiple equilibria become more likely the smaller the share parameter $a$ is, with a threshold of 0.28 . The reason is that the ratio $(1-a) / a$ acts like a shifter of the demand for non-tradables, $p=(1-a) / a\left(c^{N} / c^{T}\right)^{-1 / \xi}$. The larger the home bias $1-a$ is, the larger the shifter will be. This means that as the home bias increases so does the sensitivity of $p$ with respect to desired absorption. Finally, as shown in the bottom-right panel of the figure multiplicity of equilibrium appears to be relatively insensitive to changes in the world interest rate, $r$. Overall, the results presented in Figure 5 suggest that multiple equilibria exist for empirically plausible parameter configurations.

Figure 6 displays the policy function of debt for two economies, on with $S(\tilde{d} ; \tilde{d})>1$, left panel, and the other with $S(\tilde{d} ; \tilde{d})<1$, right panel. In both economies $\kappa=0.33, a=0.31, r=0.04$, and $y^{T}=y^{N}=1$. In the economy shown in the left panel $\xi$ is equal to 0.5 , which yields $S(\tilde{d} ; \tilde{d})=1.37$, and in the economy shown in the right panel $\xi$ is equal to 0.83 , which yields $S(\tilde{d} ; \tilde{d})=0.85$. In the economy with $\xi=0.5$, there is a range of initial debt levels, $d_{0} \in\left[\hat{d}_{0}, \tilde{d}\right]$, for which there exists more than one equilibrium value of debt in period $1, d_{1}$. In this case, one can define three equilibrium selection criteria. Let equilibrium selection criterion (A) be one in which a steady state equilibrium occurs if possible ( $d_{1}=d_{0}$ ). Under this equilibrium selection criterion, the policy function is the 45 -degree line for $d_{0} \leq \tilde{d}$ and the locus $a a^{\prime}$ for $d_{0}>\tilde{d}$. Note that when $d_{0}>\tilde{d}$, a steady-state equilibrium does not exist, because it would imply a violation of the collateral constraint. In this case, the equilibrium inevitably requires deleveraging and a binding collateral constraint in period
$0 .{ }^{6}$ Thus, under equilibrium selection criterion (A), the policy function exhibits a discontinuity at $d_{0}=\tilde{d}$. Let equilibrium selection criterion (B) be one in which the collateral constraint binds in period 0 , if possible. If two equilibrium values of $d_{1}$ exist for which the collateral constraint binds, then pick the larger one. Under this equilibrium selection criterion, the policy function is the 45 -degree line for initial debt levels $d_{0}<\hat{d}_{0}$, the locus $b b^{\prime}$ for $d_{0} \in\left[\hat{d}_{0}, \tilde{d}\right]$, and the locus $a a^{\prime}$ for $d_{0}>\tilde{d}$. Thus the policy function under equilibrium selection criterion (B) exhibits two points of discontinuity, $\hat{d}_{0}$ and $\tilde{d}$. Finally, let equilibrium selection criterion (C) be one in which the collateral constraint binds in period 0 , if possible. If two equilibrium values of $d_{1}$ exist for which the collateral constraint binds, then pick the smaller one. The policy function then is the 45 -degree line for $d_{0}<\hat{d}_{0}$, and the segment $b a^{\prime}$ for $d_{0} \geq \hat{d}_{0}$. Thus, under equilibrium selection criterion (C) the debt policy function is discontinuous at point $d_{0}=\hat{d}_{0}$.

As shown in the right panel, when $\xi=0.83$ there is a unique equilibrium and the policy function is continuous. Thus, a key difference between the economies with $\xi=0.5$ and $\xi=0.83$ is that for any for the three equilibrium selection criteria considered the policy function is discontinuous in the former economy, whereas the policy function is continuous in latter. These results apply more generally. Any parameterization satisfying $S(\tilde{d}, \tilde{d})>1(<1)$ and the remaining conditions of Proposition 1 will give rise to a policy function of the type shown in the left (right) panel of Figure 6.

## 5. MULTIPLE EQUILIBRIA IN AN ECONOMY WITH FUNDAMENTAL UNCERTAINTY

The analysis thus far has assumed no fundamental uncertainty (in fact, constant endowments) and equality of the subjective and pecuniary discount rates $(\beta(1+r)=1)$. We now investigate numerically the possibility of multiple equilibria in an environment that relaxes these assumptions.

### 5.1. Equilibrium selection

The numerical solution must take a stance on how to handle the possibility of multiplicity of equilibrium of the type identified in Section 4. Failing to address this issue may result in nonconvergence of a numerical algorithm. In addition, as suggested in previous sections and as will be demonstrated quantitatively below, under multiple equilibria, a given economy may display overborrowing, no overborrowing, or underborrowing depending on what particular equilibrium agents coordinate upon.

We focus on three equilibrium selection criteria suggested by the preceding theoretical analysis: (A) If for a given current state $\left(y_{t}^{T}, y_{t}^{N}, d_{t}\right)$ there is a value of $d_{t+1}$ for which the equilibrium conditions (10)-(15) are satisfied and the collateral constraint is not binding, pick this value of $d_{t+1}$. Otherwise, pick a value of $d_{t+1}$ for which all equilibrium conditions are satisfied. ${ }^{7}$ (B) If for a given current state $\left(y_{t}^{T}, y_{t}^{N}, d_{t}\right)$ there are one or two values of $d_{t+1}$ for which all equilibrium conditions are satisfied and the collateral constraint is binding, pick the larger one. (C) If for a given current state $\left(y_{t}^{T}, y_{t}^{N}, d_{t}\right)$ there are one or two values of $d_{t+1}$ for which all equilibrium conditions are satisfied and the collateral constraint is binding, pick the smaller one. Selection criterion (A) favours non-crisis outcomes, as in point $A$ in Figure 4. Criteria (B) and (C)

[^1]TABLE 1
Calibration

| Parameter | Value | Description |
| :--- | :---: | :--- |
|  |  | Structural parameters |
| $\kappa$ | $0.32(1+r)$ | Parameter of collateral constraint |
| $\sigma$ | 2 | Inverse of intertemporal elasticity of consumption |
| $\beta$ | 0.91 | Subjective discount factor |
| $r$ | 0.04 | Annual interest rate |
| $\xi$ | $0.5,0.83$ | Elasticity of substitution between tradables and non-tradables |
| $a$ | 0.31 | Parameter of CES aggregator |
|  | Discretization of state space |  |
| $n_{y^{T}}$ | 50 | Number of grid points for lny $y_{t}^{T}$, equally spaced |
| $n_{y^{N}}$ | 50 | Number of grid points for lny $y_{t}^{N}$, equally spaced |
| $n_{d}$ | 800 | Number of grid points for $d_{t}$, equally spaced |
| $\left[\ln \underline{y}^{T}, \ln \bar{y}^{T}\right]$ | $[-0.1093,0.1093]$ | Range for logarithm of tradable output |
| $\left.\ln y^{N}, \ln \bar{y}^{N}\right]$ | $[-0.1328,0.1328]$ | Range for logarithm of non-tradable output |
| $[\underline{d}, \bar{d}]$ | $[0.2,1.02(1+\mathrm{r})]$ | Debt range |

Notes: The time unit is a year.
favour self-fulfilling equilibria, as in points $B$ and $C$ in Figure 4, respectively, with (C) favouring larger crises.

### 5.2. Calibration, driving forces, and computation

As in Section 4, we calibrate the structural parameters of the model following Bianchi (2011), with one exception. Specifically, we lower the intratemporal elasticity of substitution, $\xi$, from 0.83 to 0.5 . We do this for two reasons. First, as argued in Section 4, this value of $\xi$ is empirically appealing. Second, we showed there that for $\xi=0.5$ the model without fundamental uncertainty and no impatience displays multiple equilibria, which makes this case a good candidate for investigating the possibility of multiplicity of equilibrium in the present environment. For comparison purposes, we will also consider the case $\xi=0.83$. The top panel of Table 1 summarizes the calibration of the structural parameters.

As in Bianchi (2011), we assume that the exogenous variables $y_{t}^{T}$ and $y_{t}^{N}$ follow the bivariate AR(1) process:

$$
\left[\begin{array}{l}
\ln y_{t}^{T}  \tag{27}\\
\ln y_{t}^{N}
\end{array}\right]=\left[\begin{array}{cc}
0.901 & -0.453 \\
0.495 & 0.225
\end{array}\right]\left[\begin{array}{l}
\ln y_{t-1}^{T} \\
\ln y_{t-1}^{N}
\end{array}\right]+\epsilon_{t},
$$

where $\epsilon_{t} \sim N\left(\emptyset, \Sigma_{\epsilon}\right)$ with $\Sigma_{\epsilon}=\left[\begin{array}{lll}0.00219 & 0.00162 \\ 0.00162 & 0.00167\end{array}\right]$.
To approximate the equilibrium, we use an Euler equation iteration procedure over a discretized state space. The bottom panel of Table 1 provides information about the discretization of the state space. The economy possesses two exogenous states, $y_{t}^{T}$ and $y_{t}^{N}$, and one endogenous state, $d_{t}$. The bounds of the endowment grids are $\left[\ln \underline{y}^{T}, \ln \bar{y}^{T}\right]=[-0.1093,0.1093]$ and $\left[\ln \underline{y}^{N}, \ln \bar{y}^{N}\right]=[-0.1328,0.1328]$ and are taken from Bianchi (2011). We discretize $\ln y_{t}^{T}$ and $\ln y_{t}^{N}$ using 50 evenly spaced points each centred at 0 . We use a finer grid than the one considered by Bianchi to increase the accuracy of the numerical approximation. However, the


Figure 7
Unconditional distribution of debt, $\xi=0.5$
Notes; Replication program plotd.min sgu_res2020.zip.
results reported below also obtain for the coarser grid used in that paper. We use the simulation approach of Schmitt-Grohé and Uribe (2009) to construct the transition probability matrix. ${ }^{8}$

For the endogenous state variable, $d_{t}$, we use 800 equally spaced points in the interval $[\underline{d}, \bar{d}]=$ $[0.2,1.02(1+r)]$. To avoid clutter, the displayed debt densities are smoothed out as follows. For each grid point $d_{i}$ the associated smoothed density is the average of the densities associated with grid points $d_{i-20}$ to $d_{i}$ for $i=21, \ldots, 800$. Thus the smoothed density has 780 bins.

### 5.3. Numerical results

Figure 7 displays the unconditional distribution of external debt, $d_{t}$ for the case that $\xi=0.5$. The different equilibrium selection criteria give rise to different debt distributions, revealing the presence of multiple equilibria. The more pessimistic equilibrium selection criterion (C), yields a debt distribution lying to the left of the others. The distribution of debt associated with selection criterion (A), which avoids binding collateral constraints when possible, lies to the right of the other distributions. The debt distribution associated with equilibrium selection criterion (B), which favours smaller debt crises, is located in between the distributions associated with criteria (A) and

[^2](C). The average debt-to-output ratio, $\left[d_{t+1} /(1+r)\right] /\left(y_{t}^{T}+p_{t} y_{t}^{N}\right)$, is $27.9 \%, 27.5 \%$, and $26.9 \%$, for criteria (A), (B), and (C), respectively. Although these debt differences may seem small, they are of the same order of magnitude as the debt difference of $0.6 \%$ of output reported by Bianchi (2011) between the unregulated economy and the Ramsey optimal one ( $29.2 \%$ versus $28.6 \%$ of output), which served as the measure of overborrowing in that paper.

Importantly, the three selection criteria deliver significant differences in the frequency of financial crises, defined as a period with a binding collateral constraint. On average, the economy suffers a financial crisis every 16 years under criterion (B), every 40 years under criterion (C), and every 71 years under criterion (A). The reason why crises are less frequent under equilibrium selection criterion (A) is that by definition this criterion avoids crises if possible. The reason why crises are less frequent under criterion (C) than under criterion (B) is that crises under criterion (C) are on average deeper in the sense that they require larger deleveraging. As a result an economy that emerges from a crisis of type (C) is less indebted and therefore less prone to financial crises in the future. In other words, deep financial crises have a cleansing effect that strengthen economic fundamentals. As we will see shortly, the frequency of financial crisis is a key dimension along which the unregulated economies (A), (B), and (C), differ from the Ramsey optimal economy.

What do the predicted self-fulfilling financial crises look like? Figures 8 and 9 display the typical self-fulfilling financial crisis of type (B) and (C), respectively. Self-fulfilling crises are identified as follows. We first simulate the economy under equilibrium selection criteria (B) or (C) for two million periods and extract all crisis episodes, defined as a binding collateral constraint. For each crisis, we construct an 11-year time window centred around the crisis. Inside the window time is normalized to run from year -5 to year 5 . Not all of these crises can be classified as self-fulfilling. Some of them are inevitable and driven purely by movements in fundamentals. To isolate the crises that could have been avoided (the self-fulfilling ones), for each crisis we construct the paths of debt and collateral that would have obtained under equilibrium selection criterion (A) by setting the state $\left(y^{T}, y^{N}, d\right)$ at the observed value at the beginning of the window, $t=-5$, and feeding the path of the exogenous state, $\left(y^{T}, y^{N}\right)$ for $t=-4,-3, \ldots, 5$. If the collateral constraint does not bind under criterion (A) at any point in the window, then the crisis is judged to be self-fulfilling. We find that $67 \%$ of the crises under criterion (B) and $46 \%$ of the crises under criterion (C) are self-fulfilling. For comparison, Figures 8 and 9 display with broken lines the corresponding paths under equilibrium selection criterion (A).

The main message conveyed by both figures is that a self-fulfilling crisis has all the characteristics of a typical sudden stop (Calvo et al., 2004): a sharp contraction in domestic absorption, an improvement in the current account, a depreciation of the real exchange rate, deleveraging, and a collapse in the value of collateral. In line with the analytical results of Section 4, the typical self-fulfilling crisis takes place in the context of weak fundamentals. At the time of the crisis, the economy undergoes a recession driven by negative output shocks. Pessimistic sentiments then aggravate the recession, making the contraction in aggregate demand and the fall in the value of collateral more pronounced. By contrast, when agents do not act on their fears, the economy behaves more along the lines of the intertemporal approach to the current account. Because the output process is persistent, the recession does cause a contraction in aggregate demand, but because the collateral constraint remains slack, agents are not forced to deleverage.

The Ramsey optimal allocation is relatively straightforward to compute because it can be cast in the form of the following Bellman equation problem:

$$
v\left(y^{T}, y^{N}, d\right)=\max _{\left\{c^{T}, d^{\prime}\right\}}\left\{U\left(A\left(c^{T}, y^{N}\right)\right)+\beta \mathbb{E}\left[v\left(y^{T^{\prime}}, y^{N^{\prime}}, d^{\prime}\right) \mid y^{T}, y^{N}\right]\right\}
$$



Figure 8
Typical self-fulfilling crisis: equilibrium selection criterion (B)
Notes: Replication program self_fulfilling_b.min sgu_res2020.zip.


Figure 9
Typical self-fulfilling crisis: equilibrium selection criterion (C)
subject to

$$
c^{T}+d=y^{T}+\frac{d^{\prime}}{1+r}
$$

and

$$
d^{\prime} \leq \kappa\left[y^{T}+\frac{1-a}{a}\left(\frac{c^{T}}{y^{N}}\right)^{\frac{1}{\xi}} y^{N}\right]
$$

where a prime superscript denotes next-period values. Although the constraints of this control problem may not represent a convex set in tradable consumption and debt, the fact that the optimal allocation is the result of a utility maximization problem, implies that its solution is generically unique.

Figure 7 displays with a solid line the unconditional distribution of net external debt, $d_{t}$, under the Ramsey optimal policy. The debt distribution is located to the right of the distribution associated with equilibrium selection criterion (C), to the left of the distribution associated with criterion (A) and at a similar position as the distribution associated with criterion (B). The average debt-to-output ratio under the Ramsey policy is $27.6 \%$. Thus the unregulated economy underborrows under equilibrium selection criterion (C) and overborrows under selection criterion (A). The amount of underborrowing under selection criterion (C) is $0.7 \%$ of output, which is comparable with the amount of overborrowing documented in Bianchi (2011), 0.6\% of output. Quantitatively, the difference between the Ramsey allocation and the three unregulated equilibrium allocations is most striking with respect to the frequency of financial crisis. Under the Ramsey optimal allocation a financial crisis occurs once every 270 years on average. This means that the Ramsey planner manages to virtually eliminate the risk of a financial crisis.

It is of interest to compare the quantitative results discussed here with those that obtain when $\xi$ is set at 0.83 , the value assumed in Bianchi (2011). Figure 10 displays the unconditional distribution of debt in the competitive equilibrium and in the Ramsey allocation under this calibration. ${ }^{9}$ Recall that this parameter setting is the only difference between the present economy and the one studied in that paper. The figure replicates the overborrowing result stressed by Bianchi. Applying equilibrium selection criteria (A), (B), or (C), we obtain the same equilibrium allocation, which suggests that the equilibrium is unique. As in the case of $\xi=0.5$, the Ramsey policy results in a significant reduction in the frequency of a financial crisis. The collateral constraint binds once every 10 years in the unregulated economy and only once every 36 years in the Ramsey economy.

Figures 11 and 12 display the debt policy functions in the competitive equilibrium and under the Ramsey allocation in the economies with $\xi=0.5$ and $\xi=0.83$, respectively. In the figures, the exogenous states, $y^{T}$ and $y^{N}$, are set at their respective means. All policy functions have a remarkable resemblance to the corresponding ones in the perfect-foresight economy studied in Section 4 (Figure 6). In particular, the key difference between the policy functions in the economy with $\xi=0.5$ and the economy with $\xi=0.83$ continues to be that in the former the policy functions are discontinuous for all three equilibrium selection criteria considered as well as for the Ramsey allocation, whereas in the latter the policy functions are continuous. The multiplicity of equilibrium in the economy with $\xi=0.5$ is reflected in the fact that the debt policy functions are different under the different equilibrium selection criteria. By contrast, when $\xi=0.83$ all equilibrium selection criteria give rise to the same policy function. One property of the
9. As in the economy with $\xi=0.5$, the grid for the exogenous states, $y_{t}^{T}$ and $y_{t}^{N}$, contains 2189 pairs compared to 16 in Bianchi's approximation. Results are robust to using Bianchi's discretization of the exogenous state space.


Figure 10
Unconditional distribution of debt, $\xi=0.83$
Notes: Replication program plotd_xi83.min sgu_res2020.zip.
calibration with $\xi=0.83$ is that the slope condition for multiplicity of equilibrium, $S(\tilde{d} ; \tilde{d})>1$, given in Proposition 1, is not satisfied for any pair $\left(y_{t}^{T}, y_{t}^{N}\right)$ in the discretized state space. By contrast, when $\xi=0.5$, the slope condition is satisfied for all pairs. This suggests that checking the slope condition for all nodes in the exogenous state space provides a useful diagnostic test for the presence of multiplicity of equilibrium. Implementing this test is straightforward and may help to avoid non-convergence problems of numerical algorithms for the approximation of the competitive equilibrium in the class of economies studied in this article.

## 6. SUNSPOTS AND PERSISTENT FINANCIAL CRISES

In the perfect-foresight economy studied in Section 4, self-fulfilling financial crises last for only one period. Multi-period crises equilibria do not exist. To see this, suppose, on the contrary, that the economy suffers self-fulfilling financial crises in periods $t$ and $t+1$. Use equation (17) to replace $y_{t}^{T}+\frac{d_{t+1}}{1+r}-d_{t}$ by $c_{t}^{T}$ in equation (18) holding with equality and solve for $c_{t}^{T}$ as an increasing function of $d_{t+1}$ :

$$
\begin{equation*}
c_{t}^{T}=\left[\left(\frac{d_{t+1}}{\kappa}-y^{T}\right) \frac{a}{1-a}\right]^{\xi} . \tag{28}
\end{equation*}
$$



Figure 11
Debt policy functions with fundamental uncertainty, $\xi=0.5$
Notes: The exogenous states $y^{T}$ and $y^{N}$ are set at their respective means. Replication program plot_policy_dp.m in sgu_res2020.zip.

A similar expression holds in $t+1$, since the collateral constraint binds in that period as well,

$$
c_{t+1}^{T}=\left[\left(\frac{d_{t+2}}{\kappa}-y^{T}\right) \frac{a}{1-a}\right]^{\xi}
$$

From the analysis of previous sections, we know that if the economy is in a financial crisis in $t+1$, it deleverages, that is $d_{t+2}<d_{t+1}$. This implies, from the above two equations, that $c_{t+1}^{T}<c_{t}^{T}$. But this is impossible in equilibrium, because according to the Euler equation (16), it would require $\mu_{t}<0$, violating the non-negativity requirement of this Lagrange multiplier, equation (20).

The predicted one-period life of financial crises is at odds with observed episodes of financial duress, which are typically multi-period phenomena. In this section, we show that in a setting with non-fundamental uncertainty self-fulfilling financial crises can be persistent. To establish this result, we characterize a two-period self-fulfilling financial crisis. The analysis, however, can be extended to longer lasting crises. Assume that $\xi \in(0,1)$. The economy is the same infinitehorizon environment studied in Section 4, with one modification. Suppose there is an exogenous random variable $s_{t}$ that takes on the values 1 or 0 . If $s_{t}$ takes the value 1 , then consumers feel


Figure 12
Debt policy functions with fundamental uncertainty, $\xi=0.83$
Notes: The exogenous states $y^{T}$ and $y^{N}$ are set at their respective means. Replication program plot_policy_dp_xi83.m in sgu_res2020.zip.
pessimistic, and if $s_{t}$ takes on the value 0 , then agents have an optimistic outlook. The variable $s_{t}$ is known as a sunspot because its sole role is to coordinate agents' expectations.

The economy starts with pessimistic sentiments, so that $s_{0}=1$. In period $1, s_{t}$ takes the value 1 with probability $\pi$ and the value 0 with probability $1-\pi$, where $\pi \in(0,1)$ is a parameter. Suppose that pessimism lasts for at most 2 periods, so that $s_{t}=0$ for all $t \geq 2$. We wish to show that there exists a probability distribution of $s_{1}$, that is, a value of $\pi$, that can support a two-period self-fulfilling financial crisis as a rational expectations equilibrium. We define a two-period selffulfilling financial crisis equilibrium as an equilibrium in which the collateral constraint binds in periods 0 and 1 . We focus on equilibria in which the economy reaches a steady state in period 2 . We establish this result by construction.

The level of debt in period 1 is determined by the collateral constraint (18) holding with equality, that is,

$$
d_{1}=\kappa\left[y^{T}+\frac{(1-a)}{a}\left(y^{T}+\frac{d_{1}}{(1+r)}-d_{0}\right)^{\frac{1}{\xi}}\right] .
$$

From Section 4, we know that this equation yields a positive real value of $d_{1}$ under the assumptions that $S(\tilde{d} ; \tilde{d})>1$ and $d_{0} \in\left(\hat{d}_{0}, \tilde{d}\right)$, which we maintain. Furthermore, the analysis presented in Section 4 shows that the economy deleverages in period 0 , that is,

$$
d_{1}<d_{0}
$$

Consumption is guaranteed to be positive (by the assumption $d_{0} \in\left(\hat{d}_{0}, \tilde{d}\right)$ ) and given by the resource constraint (17)

$$
c_{0}^{T}=y^{T}+\frac{d_{1}}{1+r}-d_{0} .
$$

In period 1, the equilibrium levels of debt and consumption depend on the realization of the sunspot variable $s_{1}$. Let $d_{t+1, i}$ and $c_{t, i}^{T}$ denote the levels of debt and consumption for $t \geq 1$ if $s_{1}=i$ for $i=0,1$.

If $s_{1}=0$, then the economy reaches a steady state with $d_{t+1,0}=d_{1}$ and $c_{t, 0}^{T}=c_{1,0}^{T}$, for all $t \geq 1$, where

$$
c_{1,0}^{T}=y^{T}-\frac{r}{1+r} d_{1} .
$$

The above three expressions imply that

$$
c_{1,0}^{T}>c_{0}^{T}
$$

If $s_{1}=1$, the economy experiences a self-fulfilling financial-crisis equilibrium in period 1 , with a binding collateral constraint in period 1 and a steady state starting in period 2. From the analysis presented in Section 4, we know that such an equilibrium exists if $d_{1}>\hat{d}_{0}$. This will be the case if $d_{0}$ is sufficiently close to $\tilde{d}$. Furthermore, since when $s_{1}=1$, the collateral constraint binds in periods 0 and 1 , we have, from the analysis at the beginning of this section, that consumption must decline between periods 0 and 1 , that is,

$$
c_{1,1}^{T}<c_{0}^{T}
$$

This construction guarantees that all equilibrium conditions (equations (16)-(21)) are satisfied for all $t \geq 0$ and $s_{1}=0,1$, except for the Euler equation (16) in period 0 . Thus, as the final step of this proof, we show that one can pick $\pi$ to ensure that the Euler equation holds in period 0 . This equation is given by

$$
\left[1-(1+r) \mu_{0}\right] \Lambda\left(c_{0}^{T}\right)=\pi \Lambda\left(c_{1,1}^{T}\right)+(1-\pi) \Lambda\left(c_{1,0}^{T}\right)
$$

The variables $c_{0}^{T}, c_{1,0}^{T}$, and $c_{1,1}^{T}$ can be taken as constants in this expression, because we have already established that they are uniquely determined by $d_{0}$ independently of $\pi$ or $\mu_{0}$. Thus, the existence of this equilibrium hangs on the existence of values of $\pi \in(0,1)$ that guarantee satisfaction of this Euler equation for a non-negative value of $\mu_{0}$. Such values do exist, because $c_{1,1}^{T}<c_{0}^{T}<c_{1,0}^{T}$ and because $\Lambda(\cdot)$ is a decreasing function. In fact, there is a range of values of $\pi$ that make $\mu_{0} \geq 0$, which is given by

$$
\pi \in\left(0, \pi^{*}\right]
$$

where

$$
\begin{equation*}
\pi^{*} \equiv \frac{\Lambda\left(c_{0}^{T}\right)-\Lambda\left(c_{1,0}^{T}\right)}{\Lambda\left(c_{1,1}^{T}\right)-\Lambda\left(c_{1,0}^{T}\right)} \in(0,1) \tag{29}
\end{equation*}
$$

According to this expression, in order for the possibility that the financial crisis extend for two periods, it is necessary that households assign a sufficiently high probability (greater than $1-\pi^{*}$ ) to the event that the economy will emerge from the crisis in the second period $(t=1)$. Moreover, the higher the chances households place on getting out of the crisis in period 1 , the more easily the conditions for a two-period crisis to exist are satisfied. This might seem paradoxical, but is indeed intuitive. If the probability of a two-period crisis, $\pi$, is too high, then consumption is expected to decline over time. Households dislike such an allocation as they prefer a smooth path for consumption. Further, households can always avoid a declining path of consumption by saving, which they can do unrestrictedly. Therefore, an equilibrium with a declining expected path of consumption cannot exist. A sufficiently small value of a crisis, $\pi$, makes the expected path of consumption non-declining, and therefore, makes a two-period crisis possible.

## 7. OPTIMAL TAX POLICY AND IMPLEMENTATION

The pecuniary externality created by the presence of the relative price of non-tradables in the collateral constraint induces an allocation that is in general suboptimal. This externality creates room for welfare-improving policy intervention. Bianchi (2011) and Benigno et al. (2016) show that capital control policy can internalize the pecuniary externality, in the sense that it can induce the representative household to behave as if it understood that its own borrowing choices influence the relative price of non-tradables and therefore the value of collateral. Benigno et al. (2016) further show that consumption taxes can support the first best equilibrium. In this section, we show that although optimal capital control or consumption-tax policies can support the optimal allocation, they may fail to implement it, in the sense that they are also consistent with other equilibrium allocations that are suboptimal. We then characterize capital control and consumption tax policies that implement the Ramsey allocation.

### 7.1. Implementation through capital controls

Following Bianchi (2011) and Benigno et al. (2016), let $\tau_{t}$ be a proportional tax on the debt acquired in period $t$. The revenue from capital control taxes is given by $\tau_{t} d_{t+1} /(1+r)$. The government is assumed to consume no goods and to rebate all revenues from capital controls to the public in the form of lump-sum transfers, denoted $\ell_{t} .{ }^{10}$ The household's sequential budget constraint now becomes

$$
c_{t}^{T}+p_{t} c_{t}^{N}+d_{t}=y^{T}+p_{t} y^{N}+\left(1-\tau_{t}\right) \frac{d_{t+1}}{1+r}+\ell_{t}
$$

All equilibrium conditions in the economy with capital control taxes are the same as in the economy without capital control taxes, equations (16)-(21), but for the Euler equation (16), which now becomes

$$
\begin{equation*}
\Lambda\left(c_{t}^{T}\right)\left[\left(1-\tau_{t}\right)-(1+r) \mu_{t}\right]=\Lambda\left(c_{t+1}^{T}\right) \tag{30}
\end{equation*}
$$

Suppose that the parameterization of the model is such that $S(\tilde{d} ; \tilde{d})>1$ and that $d_{0} \in\left(\hat{d}_{0}, \tilde{d}\right)$, so that self-fulfilling financial-crisis equilibria exist, as shown in Figure 4. Since one possible competitive equilibrium is $d_{t}=d_{0}$ for all $t$ (point $A$ in Figure 4) with $c_{t}^{T}=y^{T}-r d_{0} /(1+r)$ for all $t \geq 0$, and since this equilibrium is the first best equilibrium (i.e. the equilibrium that would result in the absence of the collateral constraint), it also has to be the optimal competitive equilibrium. The capital control tax associated with the optimal equilibrium is zero at all times. This can be deduced from inspection of equation (30). Because the equilibrium associated with point A is a steady state, consumption of tradables is constant over time. And, because at point A the collateral constraint is slack, $\mu_{t}=0$ for all $t$. It follows that $\tau_{t}=0$ for all $t \geq 0$.

The optimal capital control policy, $\tau_{t}=0$, does not guarantee that the competitive equilibrium will be the optimal one (point A in Figure 4). In particular, the policy rule $\tau_{t}=0$ for all $t$ may result in an unintended competitive equilibrium, like points B or C in Figure 4. Thus, the optimal policy setting, $\tau_{t}=0 \forall t$, may fail to implement the optimal allocation. However, any capital control policy that succeeds in implementing the optimal allocation must deliver $\tau_{t}=0$ for all $t$ in equilibrium. The difference between a policy that sets $\tau_{t}=0$ under all circumstances and a

[^3]policy that implements the optimal allocation does not lie in the capital control tax that results in equilibrium, but in the tax rates that would be imposed off the optimal equilibrium.

To shed light on the issue of implementation, here we study a capital control feedback rule that implements the optimal equilibrium. Specifically, consider the capital control policy

$$
\begin{equation*}
\tau_{t}=\tau\left(d_{t+1}, d_{t}\right) \tag{31}
\end{equation*}
$$

satisfying $\tau(d, d)=0$. To see whether this capital control policy is consistent with the optimal equilibrium, it suffices to verify that the Euler equation (30) is satisfied, since this is the only equilibrium condition in which $\tau_{t}$ appears. Under the tax-policy rule (31), the Euler equation in period 0 is given by

$$
\begin{equation*}
\frac{\Lambda\left(c_{0}^{T}\right)}{\Lambda\left(c_{1}^{T}\right)}=\frac{1}{1-\tau\left(d_{1}, d_{0}\right)-(1+r) \mu_{0}} \tag{32}
\end{equation*}
$$

In the optimal equilibrium, we have that $c_{1}^{T} / c_{0}^{T}=1$, that $d_{1}=d_{0}$ (which implies that $\tau\left(d_{1}, d_{0}\right)=0$ ), and that $\mu_{0}=0$, so the Euler equation holds. This establishes that the proposed policy is consistent with the optimal allocation.

In addition to supporting the optimal equilibrium, if appropriately designed, the tax policy (31) can rule out the unintended equilibria. Recalling that $c_{0}^{T}$ and $c_{1}^{T}$ satisfy $c_{0}^{T}=y^{T}+d_{1} /(1+r)-d_{0}$ and $c_{1}^{T}=y^{T}-r d_{1} /(1+r)$, we can write the Euler equation (32) as

$$
\begin{equation*}
\frac{\Lambda\left(y^{T}+d_{1} /(1+r)-d_{0}\right)}{\Lambda\left(y^{T}-r /(1+r) d_{1}\right)}=\frac{1}{1-\tau\left(d_{1}, d_{0}\right)-(1+r) \mu_{0}} . \tag{33}
\end{equation*}
$$

Now pick the function $\tau(\cdot, \cdot)$ in such a way that if a self-fulfilling financial-crisis equilibrium occurs and the economy deleverages, then the Euler equation holds only if $\mu_{0}$ is negative. Specifically, set $\tau\left(d_{1}, d_{0}\right)$ to satisfy

$$
\frac{\Lambda\left(y^{T}+d_{1} /(1+r)-d_{0}\right)}{\Lambda\left(y^{T}-r /(1+r) d_{1}\right)}<\frac{1}{1-\tau\left(d_{1}, d_{0}\right)}
$$

for all $d_{1}<d_{0}$. Clearly, this policy requires $\tau\left(d_{1}, d_{0}\right)>0$ if $d_{1}<d_{0}$. Under this capital control policy, the Euler equation would not hold for any value of $d_{1}$ less than $d_{0}$, since it would require $\mu_{0}<0$, which violates the non-negativity constraint (20). This means that any equilibrium in which the economy deleverages is ruled out.

The capital control policy that rules out self-fulfilling financial-crisis equilibria and ensures that only the optimal equilibrium emerges is one in which the policy maker is committed to imposing capital control taxes in the case of capital outflows, that is, in the event that $d_{1}<d_{0}$. This type of capital control policy serves as a metaphor for a variety of policies that are often contemplated in emerging countries during financial panics and that aim at temporarily restricting capital outflows, including restrictions on foreign exchange markets and profit and dividend repatriations. In the present perfect-foresight economy, the mere threat of the imposition of capital control taxes in the event of capital flights suffices to fend off self-fulfilling financial crises. In equilibrium, these threats never need to be carried out.

### 7.2. Implementation through consumption subsidies

The Ramsey optimal allocation can also be implemented by means of consumption subsidies. We focus on subsidies on non-tradable consumption, but the result also obtains when the policy instrument is a tax on tradable consumption. Let $\tau_{t}^{N}$ be a proportional subsidy on non-tradable
consumption in period $t$. It generates government spending in the amount $\tau_{t}^{N} p_{t} c_{t}^{N}$. We continue to assume that the government consumes no goods and that it finances spending in a lump-sum fashion. Its budget constraint is then given by $\tau_{t}^{N} p_{t} c_{t}^{N}=\ell_{t}$, where $\ell_{t}$ denotes lump-sum taxes. The budget constraint of the household is

$$
c_{t}^{T}+\left(1-\tau_{t}^{N}\right) p_{t} c_{t}^{N}+d_{t}=y^{T}+p_{t} y^{N}+\frac{d_{t+1}}{1+r}-\ell_{t} .
$$

All first-order conditions associated with the household optimization problem are as in the economy without taxes except for equation (6), which becomes

$$
\left(1-\tau_{t}^{N}\right) p_{t}=\frac{1-a}{a}\left(\frac{c_{t}^{T}}{c_{t}^{N}}\right)^{1 / \xi}
$$

It is clear from the analysis of Section 7.1 that a zero subsidy supports the first-best allocation (see Figure 4). This result was first shown in Benigno et al. (2016). However, although the policy $\tau_{t}^{N}=0$ supports the Ramsey optimal allocation, it does not implement it, as it leaves the door open for self-fulfilling crises to occur (equilibria B or C). Suppose instead that the government sets a constant proportional subsidy on non-tradable consumption, $\tau^{N}>0$.

The competitive equilibrium conditions in the economy with the consumption subsidy are identical to those in the economy without the subsidy, equations (16)-(21), except for the collateral constraint (18), which becomes

$$
\begin{equation*}
d_{t+1} \leq \kappa\left[y^{T}+\frac{(1-a)}{a} \frac{\left(y^{T}+\frac{d_{t+1}}{(1+r)}-d_{t}\right)^{\frac{1}{\xi}}}{1-\tau^{N}}\right] \tag{34}
\end{equation*}
$$

and for the complementary slackness condition (19), which undergoes a similar modification.
A positive $\tau^{N}$ increases the value of collateral and shifts the right-hand side of the period- 0 collateral constraint up. Thus, since the collateral constraint is always slack in the steady-state equilibrium (point A in Figure 4), it will also be slack for any positive and constant $\tau^{N}$. This means that any constant and positive $\tau^{N}$ supports the first best equilibrium. A positive value of $\tau^{N}$ also makes it less likely that financial-crisis equilibria of type B or C exist, because it shifts the right-hand side of the period- 0 collateral constraint further above the 45 -degree line. In fact, given a value of $d_{0}$, the government can set $\tau^{N}$ such that the period- 0 collateral constraint does not intersect the 45 -degree line thereby making equilibria of type B or C impossible. It can be shown that this will be the case for any $\tau^{N}$ satisfying $\left(1-\tau^{N}\right)<\left[\frac{(1+\kappa /(1+r)) y^{T}-d_{0}}{(1+\kappa /(1+r)) y^{T}-\hat{d}_{0}}\right]^{(1-\xi) / \xi}$, where $\hat{d}_{0}$ is the lower bound of the range of initial debt levels $d_{0}$ for which multiple equilibria exist and is defined in Proposition 1. The right-hand side of this expression is unambiguously less than unity if multiple equilibria exist in the absence of government intervention. This means that to rule out equilibria B or C , the government must introduce a strictly positive subsidy on non-tradable consumption.

The intuition for why a constant subsidy on non-tradable consumption is an effective instrument to eliminate unintended equilibria is as follows. Suppose economic agents for no fundamental reason become pessimistic and decide to cut spending and deleverage. The fall in aggregate demand depresses the relative price of non-tradables, which in turn reduces the value of collateral. However, the optimal subsidy puts a floor to the value of collateral so that the decline induced by the negative sentiments is insufficient to bring about a binding collateral constraint, thereby invalidating the initial pessimistic views.

## 8. CONCLUSION

In open economy models in which borrowing is limited by the value of tradable and nontradable output, the equilibrium value of collateral is increasing in end-of-period external debt. For plausible calibrations, this relationship can become perverse, in the sense that an increase in debt increases collateral by more than one for one. That is, as the economy chooses higher debt levels it becomes less leveraged. This problem can give rise to a non-convexity whereby two disjoint ranges of external debt for which the collateral constraint is satisfied are separated by a range for which the collateral constraint is violated.

This article shows that in this environment, the economy can display financial crises in which pessimistic views about the value of collateral are self-fulfilling. In a stochastic economy and under empirically plausible calibrations, there exist self-fulfilling crises that have all the characteristics of observed sudden stops: the economy deleverages, the real exchange rate depreciates sharply, and the current account improves in the context of depressed aggregate demand. These expectations-driven crises are linked to economic fundamentals as they are more likely to occur in periods of negative output or terms-of-trade shocks.

Under parameterizations for which self-fulfilling crises exist, the economy can display underborrowing, in the sense that the average equilibrium level of debt is lower than the one in the constrained optimal allocation. The underborrowing result stands in contrast to the overborrowing result stressed in the related literature. Underborrowing emerges in the present context because in economies that are prone to self-fulfilling financial crises, individual agents engage in excessive precautionary savings as a way to self-insure.

The article addresses the issue of implementation of the constrained optimal allocation. This is a non-trivial problem, because the optimal policy only specifies what taxes are levied in equilibrium, but not what taxes should be off the desired equilibrium. As a result, the optimal policy need not ensure implementation of the optimal allocation. In particular, other, possibly welfare inferior, equilibria may be consistent with the optimal policy. This article shows that a capital control tax policy that threatens to tax capital outflows in the event of a self-fulfilling financial crisis can make such events incompatible with a rational expectations equilibrium and therefore eliminate them as possible outcomes, ensuring the emergence of the desired equilibrium.

## APPENDIX

## A. PROOF OF PROPOSITION 1: EXISTENCE OF SELF-FULFILLING FINANCIAL-CRISIS EQUILIBRIA WHEN $\mathbf{0}<\xi<1$

We already established that $S(\tilde{d} ; \tilde{d})>1$ is a sufficient condition for the coexistence of a steady-state equilibrium and a self-fulfilling financial-crisis equilibrium. We now show that this condition is also necessary. Let $d_{0}=d^{\prime}$, where $d^{\prime} \in(0, \tilde{d})$. Consider the value of $d_{1}$, denoted $d_{1}^{c}$, such that the right-hand side of the period- 0 collateral constraint associated with $d_{0}=d^{\prime}$ equals $\tilde{d}$. This is point $C$ in Figure A1.

At point $C$, the slope of the period- 0 collateral constraint is equal to $S(\tilde{d} ; \tilde{d})$, that is, $S\left(d_{1}^{c} ; d^{\prime}\right)=S(\tilde{d} ; \tilde{d})$. This is because both the level and the slope of the right-hand side of the period-0 collateral constraint depend only on the value of $d_{1} /(1+r)-d_{0}$. Since the period- 0 collateral constraint is slack at point $C$, a crossing with the 45 -degree line can only occur at a value of $d_{1}$ below $d_{1}^{c}$. However, since the right-hand side of the period- 0 collateral constraint is increasing and convex in $d_{1}$, such a crossing is only possible if $S(\tilde{d} ; \tilde{d})>1$.

It remains to characterize the interval of initial values of debt, $d_{0}$, for which self-fulfilling financial-crisis equilibria coexist with steady-state equilibria. The upper bound of this interval is $\tilde{d}$ (point $X$ in Figure A2). This is because, as shown in Section 3, no steady-state equilibrium exists for $d_{0}>\tilde{d}$. Let $\hat{d}_{0}$ denote the lower bound of this interval. Because the right-hand side of the period- 0 collateral constraint (24) is a decreasing function of $d_{0}$ and an increasing and convex function of $d_{1}$, it is clear that the right-hand side of the period 0 collateral constraint associated with $d_{0}=\hat{d}_{0}$ must be tangent to the 45 -degree line. Figure A2 displays this locus. The point of tangency is $T$, with an associated value of


Figure A1
Necessary condition, $S(\tilde{d} ; \tilde{d})>1$
Notes: The broken line is the right-hand side of the period- 0 collateral constraint when $d_{0}=d^{\prime}$. The downward sloping line is the right-hand side of the steady-state collateral constraint. At point $C$ the slope of the period- 0 collateral constraint is $S\left(d_{1}^{c} ; d^{\prime}\right)=S(\tilde{d} ; \tilde{d})$. A necessary condition for the right-hand side of the period- 0 collateral constraint to intersect the 45 -degree line is that $S(\tilde{d} ; \tilde{d})>1$.
$d_{1}$ denoted $\hat{d}_{1}$. The point at which this locus crosses the right-hand side of the long-run collateral constraint (point $V$ ), determines the lower bound $\hat{d}_{0}$.

Formally, $\hat{d}_{1}$ and $\hat{d}_{0}$ are the values of $d_{1}$ and $d_{0}$ at which the period- 0 collateral constraint (24) binds and the slope of its right-hand side, given in equation (26), equals one, that is, respectively,

$$
\begin{equation*}
\hat{d}_{1}=\kappa\left[y^{T}+\left(\frac{1-a}{a}\right)\left(y^{T}+\frac{\hat{d}_{1}}{1+r}-\hat{d}_{0}\right)^{\frac{1}{\xi}}\right] \tag{A.1}
\end{equation*}
$$

and

$$
\begin{equation*}
\kappa\left(\frac{1-a}{a}\right) \frac{1}{(1+r)} \frac{1}{\xi}\left(y^{T}+\frac{\hat{d}_{1}}{1+r}-\hat{d}_{0}\right)^{\frac{1}{\xi}-1}=1 \tag{A.2}
\end{equation*}
$$

This system may have multiple solutions. The economically sensible solution is the one yielding positive tradable consumption in period $0, \hat{c}_{0}^{T} \equiv y^{T}+\hat{d}_{1} /(1+r)-\hat{d}_{0}>0$. It is unique. From (A.2), we have that

$$
\hat{c}_{0}^{T}=\left[\kappa\left(\frac{1-a}{a}\right) \frac{1}{(1+r)} \frac{1}{\xi}\right]^{\frac{\xi}{\xi-1}} .
$$

Using this solution to eliminate $y^{T}+\hat{d}_{1} /(1+r)-\hat{d}_{0}$ in equation (A.1), we obtain

$$
\hat{d}_{1}=\kappa\left\{y^{T}+\frac{(1-a)}{a}\left[\kappa\left(\frac{1-a}{a}\right) \frac{1}{(1+r)} \frac{1}{\xi}\right]^{\frac{1}{\xi-1}}\right\}
$$

And using the fact that $\hat{d}_{0}=y^{T}+\hat{d}_{1} /(1+r)-\hat{c}_{0}^{T}$, we obtain the solution

$$
\begin{equation*}
\hat{d}_{0}=\left(1+\frac{\kappa}{1+r}\right) y^{T}-(1-\xi)\left(\kappa \frac{1-a}{a} \frac{1}{1+r} \frac{1}{\xi}\right)^{\frac{1}{1-\frac{1}{\xi}}} \tag{A.3}
\end{equation*}
$$



Notes: The downward-sloping solid line is the right-hand side of the steady-state collateral constraint, given in equation (22). The upwardsloping broken lines are the right-hand side of the period-0 collateral constraint, given in equation (24), for two values of $d_{0}, \hat{d}_{0}$ and $\tilde{d}$. Multiple equilibria exist for initial debt levels, $d_{0}$, in the interval $\left[\hat{d}_{0}, \tilde{d}\right)$.

We have thus shown that for any given $d_{0} \in\left[\hat{d}_{0}, \tilde{d}\right)$ the period- 0 collateral constraint binds at a point located on the 45 -degree line and between points $T$ and $X$ in Figure A2 and that the associated allocation can be supported as an equilibrium.

## B. PROOF OF COROLLARY 1: EXISTENCE OF TWO SELF-FULFILLING FINANCIAL-CRISIS EQUILIBRIA WHEN $\mathbf{0}<\xi<1$

As explained in Section4, for there to exist two self-fulfilling financial-crisis equilibria, the second intersection of the righthand side of the period- 0 collateral constraint with the 45 -degree line must occur with a positive slope. This requirement ensures that in the second self-fulfilling financial-crisis equilibrium consumption of tradables is positive in period 0 . Because $\xi$ is less than one, the right-hand side of the period- 0 collateral constraint is convex. Thus, the right-hand side of the period- 0 collateral constraint will cross the 45 -degree line with positive slope twice only if the period- 0 collateral constraint is not violated at the level of debt, $d_{1}$, at which the right-hand side of the period- 0 collateral constraint has a slope of zero. Formally, using equation (26), we have that the right-hand side of the period-0 collateral constraint has a slope of zero when

$$
\begin{equation*}
y^{T}+\frac{d_{1}}{1+r}-d_{0}=0 \tag{B.1}
\end{equation*}
$$

Combining this expression with the period-0 collateral constraint given in equation (24), yields

$$
d_{1} \leq \kappa y^{T}
$$

Now evaluating the period- 0 resource constraint, equation (17), at $c_{0}^{T}=0$ and $d_{1}=\kappa y^{T}$, we have that the initial values of debt, $d_{0}$, for which the period- 0 collateral constraint is not violated when its right-hand side has slope 0 are given by

$$
\begin{equation*}
d_{0} \leq\left(1+\frac{\kappa}{1+r}\right) y^{T} . \tag{B.2}
\end{equation*}
$$



Notes: The downward-sloping solid line is the right-hand side of the steady-state collateral constraint, given in equation (22). The upwardsloping broken line is the right-hand side of the period-0 collateral constraint, given in equation (24).

This condition does not ensure two crossings of the right-hand side of the period-0 collateral constraint with the 45degree line. It only ensures that for any $d_{0}$ in this interval at the value of $d_{1}$ at which the associated right-hand side of the period- 0 collateral constraint has slope zero the period- 0 collateral constraint is not violated. So it does not rule out that for some values of $d_{0}$ in this interval the period- 0 collateral constraint never bind. To rule out this situation, we must impose, in addition, the restriction $d_{0} \in\left[\hat{d}_{0}, \tilde{d}\right)$ which ensures the existence of one crossing with positive slope to the left of $d_{0}$ (see Proposition 1). Thus we have that a necessary and sufficient condition for the existence of two self-fulfilling financial-crisis equilibria to exist is

$$
d_{0} \in\left[\hat{d}_{0}, \min \left(\left(1+\frac{\kappa}{1+r}\right) y^{T}, \tilde{d}\right)\right)
$$

This interval is meaningful, because, comparing the expression for $\hat{d}_{0}$ given in Proposition 1 with $\left(1+\frac{\kappa}{1+r}\right) y^{T}$, we have that $\left(1+\frac{\kappa}{1+r}\right) y^{T}>\hat{d}_{0}$. It can be shown that $\left(1+\frac{\kappa}{1+r}\right) y^{T}<\tilde{d}$ if and only if $S(\tilde{d} ; \tilde{d})>1 / \xi$.

## C. EXISTENCE OF MULTIPLE EQUILIBRIA WHEN $\boldsymbol{\xi}>\mathbf{1}$

Figure A3 illustrates the existence of a self-fulfilling financial-crisis equilibrium with an intratemporal elasticity of substitution larger than unity, $\xi>1$. When $\xi>1$, the right-hand side of the period- 0 collateral constraint (shown with a broken line) is concave in $d_{1}$, and, as a result, there is at most one self-fulfilling financial crisis equilibrium (point $B$ in the figure). Formally, we have the following result.

Proposition 2(Existence of multiple equilibria when $\xi>1$ ) Suppose $y_{t}^{N}=1, y_{t}^{T}=y^{T}>0, \beta(1+r)=1$, and $\xi>1$. Then, the steady-state equilibrium coexists with a self-fulfilling financial-crisis equilibrium if and only if $S(\tilde{d} ; \tilde{d})>1 / \xi$ and $d_{0} \in\left(\left(1+\frac{\kappa}{1+r}\right) y^{T}, \tilde{d}\right)$, where $S(\cdot ; \cdot \cdot)$ is the slope of the right-hand side of the period-0 collateral constraint defined in equation (26) and $\tilde{d}$ is defined in equation (23).

Proof: When $\xi>1$ the right-hand side of the period- 0 collateral constraint is increasing and concave in $d_{1}$. For $d_{1}=d_{0} \in$ $(0, \tilde{d})$, the right-hand side of the period- 0 collateral constraint, equation (24), lies above the 45 -degree line, that is, the collateral constraint is slack. Now, from the period-0 resource constraint (17) we have that, given a $d_{0} \in(0, \tilde{d})$, the smallest value of $d_{1}$ such that $c_{0}^{T}$ is non-negative is given by $(1+r)\left(d_{0}-y^{T}\right)$. If at $d_{1}=(1+r)\left(d_{0}-y^{T}\right)$, the right-hand side of (24) lies below the 45 -degree line, then there exists a value of $d_{1}$ in the interval $\left((1+r)\left(d_{0}-y^{T}\right), d_{0}\right)$, for which (24) holds with equality. At $d_{1}=(1+r)\left(d_{0}-y^{T}\right)$, the right-hand side of (24) is equal to $\kappa y^{T}$. Thus we need that $\kappa y^{T}<(1+r)\left(d_{0}-y^{T}\right)$. Rewrite this inequality as $(1+\kappa /(1+r)) y^{T}<d_{0}$. The question is then whether there exist any $d_{0}$ that satisfy this inequality and also $d_{0}<\tilde{d}$, that is, whether the interval $\left((1+\kappa /(1+r)) y^{T}, \tilde{d}\right)$ is non-empty.

Combining the definition of $\tilde{d}$ given in equation (23) with the resource constraint (17) and the slope of the right-hand side of the period- 0 collateral constraint given in (26) both evaluated at $\tilde{d}$, we can write

$$
\left(1+\frac{\kappa}{1+r}\right) y^{T}=\tilde{d}+\tilde{c}^{T}(1-\xi \tilde{S})
$$

where $\tilde{S} \equiv S(\tilde{d} ; \tilde{d})$ and $\tilde{c}^{T} \equiv y^{T}-\frac{r}{1+r} \tilde{d}$ is the level of tradable consumption in a steady-state with debt equal to $\tilde{d}$, which, as shown in Section 3, is strictly positive. It follows that $\left(1+\frac{\kappa}{1+r}\right) y^{T}<\tilde{d}$ (i.e. the interval is non-empty) if and only if $\tilde{S}>1 / \xi$. It follows that a self-fulfilling financial-crisis equilibrium exists if and only if $\tilde{S}>1 / \xi$. We have also shown that if this condition is met, the range of initial values of debt for which self-fulfilling crises exist is given by $d_{0} \in\left(\left(1+\frac{\kappa}{1+r}\right) y^{T}, \tilde{d}\right)$.

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[^0]:    The editor in charge of this paper was Veronica Guerrieri.

[^1]:    6. There are parameterizations for which no equilibrium exists for $d_{0}>\tilde{d}$. But this is not the case under the parameterizations considered in Figure 6.
    7. In the numerical application, picking the smallest instead of the largest value of $d_{t+1}$ for which all equilibrium conditions are satisfied resulted in the same equilibrium.
[^2]:    8. This procedure eliminates exogenous states that occur with probability zero. It resulted in 2,189 endowment pairs.
[^3]:    10. The results would be unchanged if one were to assume alternatively that revenues from capital control taxes are rebated by means of a proportional income transfer. Since tradable and non-tradable income is exogenous to the household, this transfer would be non-distorting and therefore equivalent to a lump-sum transfer.
