Physics GR6037 Professor Christ
due 09/20/2026

Problem Set #1

Don't miss the second problem on the next page.

1. Consider a two-dimensional problem in which a beam of particles moves to the right in the $+z$ direction with momentum p. The particles in the beam scatter from an array of $N+1$ atoms arranged along the x axis with coordinates $(x = jd, z = 0,)$ where j is an integer in the range $-N/2 \le j \le N/2$, d is the separation of the atoms in the vertical direction and N is even. Assume that each of the scattered particles can be represented by a wave which is the superposition of the $N+1$ circular waves:

$$\psi(x, z) = A \sum_{j=-N/2}^{N/2} e^{i\sqrt{(x-jd)^2+z^2}\,p/\hbar}.$$

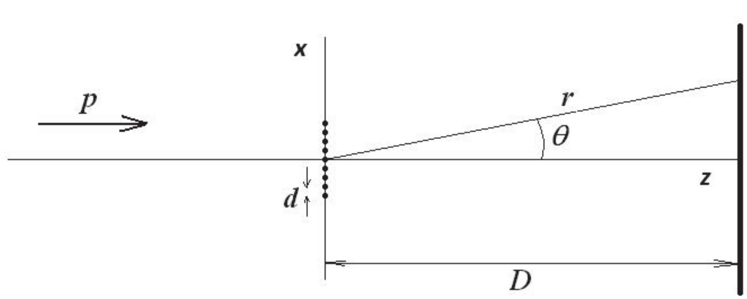
   (a) In the limit that the distance $r = \sqrt{x^2+z^2}$ is large but $\sin(\theta)$ is small, expand the exponents in the above expression for $\psi(x, z)$, retaining the term proportional to r and the term which is a constant as r becomes large.

   (b) Using this approximation, perform the sum over j and determine the resulting interference pattern that would would be seen on a screen a large distance D from the atoms as a function of θ

   (c) Finally generalize the problem to three dimensions with a third dimension y also perpendicular to the beam direction. Replace the one dimensional array of atoms by a two dimensional, attenuating screen placed placed in the x-y plane with its center at the origin. Assume that the incident wave passes through this screen with probability amplitude $\psi(x, y, 0)$ at the right-hand surface of the screen where $\psi(x, y, 0) = 0$ for $|x| > d$ or $|y| > d$ for the distance $d \ll D$. Use a Huygen's like expression for the wave function at large z simliar to the expresssion used in parts (a) and (b) above:

$$\psi(x, y, z) = A \int_{-d}^{d} dx' \int_{-d}^{d} dy' e^{i\sqrt{z^2+(x-x')^2+(y-y')^2}\,p/\hbar}\,\psi(x', y', 0).$$

   Making the same approximations as above, show that the probability of finding a particle at the position (x, y, D), $|\psi(x, y, D)|^2$, is proportional to the square of the modulus of the Fourier transform of $\psi(x, y, 0)$, evaluated at the $(p_x, p_y) = p(\frac{x}{D}, \frac{y}{D})$. This expression for (p_x, p_y) gives precisely the x and y momentum components that a particle with $p_z \approx p$ would require to reach the point (x, y, D).



2. Consider a particle moving in one dimension and located at the position x where x lies in the interval $[0, L]$. If $\psi(x)$ is the particle's wave function, assume that $\psi(x)$ obeys periodic boundary conditions:

$$\psi(0) = \psi(L) \quad \text{and} \quad \left.\frac{d}{dx}\psi(x)\right|_{x=0} = \left.\frac{d}{dx}\psi(x)\right|_{x=L}$$

(a) Consider the wave functions

$$\phi_n(x) = \frac{1}{\sqrt{L}} e^{ip_n x/\hbar}$$

where $p_n = 2\pi\hbar n/L$ and $n \in \mathbb{Z}$ is an integer. Show that the functions $\phi_n(x)$ obey:

$$\int_0^L dx\, \phi^*_{n'}(x)\phi_n(x) = \begin{cases} 1 & \text{for } n' = n \\ 0 & \text{for } n' \neq n \end{cases}$$

(b) If the wave function $\psi(x)$ can be written as the sum

$$\psi(x) = \sum_{n\in\mathbb{Z}} a_n \phi_n(x)$$

where the coefficients $\{a_n\}_{-\infty<n<\infty}$ are complex numbers, derive a formula that can be used to determine the coeficient a_n from the function $\psi(x)$.

(c) What condition must the coefficients a_n obey if the function $\psi(x)$ is normalized to 1?

3. Consider a linear operator O acting on a Hilbert space $\mathcal{H}$. Define the *kernel* of O as the subspace of $\mathcal{H}$ which is mapped to the zero vector by the operator O.

(a) Show that the kernel of O is a vector space.

Let $O^\dagger$ be the hermitian conjugate of O. (Recall $O^\dagger$ is defined by the relation $(A, O^\dagger B) = (OA, B)$ for all vectors $A, B \in \mathcal{H}$.)

(b) If the dimension of $\mathcal{H}$ is finite, show that:

$$\dim(\ker O) = \dim(\ker O^\dagger).$$

(The rank-nullity theorem.)

(c) Construct a counter example to this statement if the dimension of $\mathcal{H}$ is infinite.