

Problem Set #2

4. Consider a 3-dimensional Hilbert space and the operator O represented by the matrix:

$$O = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

- (a) Find the eigenvectors and eigenvalues of O .
 (b) What is the probability of finding each of these eigenvalues if the quantity O is measured for the state vector:

$$|\psi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}?$$

5. Consider a hermitian operator O acting on a complex vector space of finite dimension N . Consider a “sphere” of states $|\psi\rangle$ with unit norm, $\| |\psi\rangle \|^2 = 1$. Make the reasonable assumption that there should exist a vector $|\psi_0\rangle$ which lies on this sphere for which the expression,

$$\left(|\psi_0\rangle, O|\psi_0\rangle \right) = \langle \psi_0 | O | \psi_0 \rangle \quad (1)$$

takes a value less than or equal to the value of $\langle \psi | O | \psi \rangle$ found for any state $|\psi\rangle$ on this sphere. (Prove the truth of this assumption if you wish.) Consider a second unit vector $|\delta\psi\rangle$ which lies in the $N - 1$ dimensional space of vectors perpendicular to $|\psi_0\rangle$:

$$\left(|\delta\psi\rangle, |\psi_0\rangle \right) = \langle \delta\psi | \psi_0 \rangle = 0. \quad (2)$$

- (a) Define the vector

$$|\psi_\epsilon\rangle = \frac{1}{\sqrt{1 + |\epsilon|^2}} \left(|\psi_0\rangle + \epsilon |\delta\psi\rangle \right) \quad (3)$$

where ϵ is a complex number. Show that the state $|\psi_\epsilon\rangle$ also has unit length.

- (b) Since $|\psi_0\rangle$ is a state of unit length which minimizes the expectation value of O among those states of unit length we can conclude that

$$\left(|\psi_\epsilon\rangle, O|\psi_\epsilon\rangle \right) \geq \left(|\psi_0\rangle, O|\psi_0\rangle \right). \quad (4)$$

Show that if evaluated to first order in the small complex number ϵ this inequality implies that for all states $|\delta\psi\rangle$

$$\left(|\delta\psi\rangle, O|\psi_0\rangle \right) = 0. \quad (5)$$

- (c) Explain how this result implies that $O|\psi_0\rangle$ must be proportional to $|\psi_0\rangle$ or

$$O|\psi_0\rangle = \lambda_0 |\psi_0\rangle, \quad (6)$$

if the proportionality constant is labeled λ_0 , proving O has at least one eigenvector.

- (d) Consider the $N - 1$ dimensional space of vectors that are orthogonal to $|\psi_0\rangle$ and repeat the above argument to find a second eigenvector $|\psi_1\rangle$ of O .
 (e) Repeating the above procedure, deduce that any hermitian O acting on a finite dimensional vector space of dimension N can be diagonalized.

6. If $E_F(\lambda)$ is a resolution of the identity for the self-adjoint operator F : $F = \int \lambda dE_F(\lambda)$,
- (a) Show that $\langle \psi | E(\lambda) | \psi \rangle$ is a non-decreasing, non-negative function of λ .
 - (b) Find the resolution of the identity, $E_{F^2}(\lambda)$ for the operator F^2 in terms of $E_F(\lambda)$. [Although not very difficult, this part does require careful thought.]
7. Consider a family of linear operators O_α acting on a Hilbert space $\mathcal{H}$. Assume that the operator $O_\alpha O_\alpha^\dagger$ has a complete set of eigenvectors and eigenvalues which depend continuously on the parameter α . Show that the index of O_α ,

$$\text{ind}(O_\alpha) \equiv \dim(\ker O_\alpha) - \dim(\ker O_\alpha^\dagger),$$

is independent of α .

[Hint: consider the operators $O_\alpha O_\alpha^\dagger$ and $O_\alpha^\dagger O_\alpha$ and show that they have one-to-one corresponding, non-zero eigenvalues and eigenvectors, so that as α varies continuously any change in $\dim(\ker O_\alpha)$ will be matched by a change in $\dim(\ker O_\alpha^\dagger)$. Central to this problem is recognizing how the dimension of the kernel of O_α can change discontinuously from n to $n \pm 1$ as α is continuously varied.] [This invariance of the index of O under continuous changes in O is mathematically profound and underlies the connection between the index of O and geometric properties of the space on which O acts, leading to the Atiyah-Singer index theorem and instantons in QCD.]