

Problem Set #3

8. Consider two self-adjoint operators A and B . Assume that spectrum of both A and B is discrete with a corresponding complete sets of eigenvectors:

$$\begin{aligned} A|n\rangle_A &= a_n|n\rangle_A \\ B|n\rangle_B &= b_n|n\rangle_B. \end{aligned}$$

If A and B have a vanishing commutator, $[A, B] = 0$, show that A and B can be simultaneously diagonalized, *i.e.* there exists a common set of eigenvectors for both operators for each of the following situations of increasing difficulty:

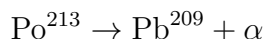
- (a) The spectrum of A and B is non-degenerate, *i.e.* for each eigenvalue of A there exists only a single eigenvector of A with that eigenvalue.
 - (b) All degeneracies are finite dimensional, *i.e.* for each eigenvalue of A there exists only a finite number of independent eigenvectors of A with that eigenvalue. [Hint: here you can use the result of problem 5 from Problem Set #2 to diagonalize a finite, hermitian matrix.]
 - (c) The case where infinite numbers of eigenvectors with the same eigenvalue are allowed. [Hint: In contrast with 6.b) we can not use problem 5 to argue that we can diagonalize an infinite-dimensional hermitian matrix. This difficulty might be avoided by applying the projection operator $P_{b_n}^B$ onto the known, discrete eigenvectors of A . Here $P_{b_n}^B$ projects onto the infinite-dimensional subspace spanned by the set of eigenvectors $\{|b_n, j\rangle_B\}_j$ of B with the same eigenvalue b_n .]
9. Prove the Schwartz inequality:

$$|\langle A|B\rangle|^2 \leq \langle A|A\rangle\langle B|B\rangle.$$

[Hint: you might use a method similar to our proof of the uncertainty relation from the commutator $[x, p] = i\hbar$ by examining the square of the norm of the vector $|A\rangle + \alpha|B\rangle$ as a function of the complex variable α and requiring that it always be non-negative.]

10. Using very approximate uncertainty principle arguments, determine:

- (a) The size of the hydrogen atom in terms of the electronic charge and mass.
- (b) The spread of α -particle energies observed in the decay:



if the Po^{213} nucleus has a lifetime of 4×10^{-6} seconds.

- (c) The maximum length of time a pencil can be balanced on its point. (Assume the pencil has a perfect point and that it is no longer balanced when it makes a noticeable angle, perhaps 10-15°, with the vertical.)

11. If $\Delta x \Delta p = \frac{\hbar}{2}$ for a state ψ , show that $\psi(x)$ must be a Gaussian wave function.
[Hint: This can be shown by returning to the derivation of the uncertainty relation given in class and examining the implications for $\psi(x)$ of the limiting case $\Delta x \Delta p = \frac{\hbar}{2}$.]
12. If the commutator $[A, B]$ itself commutes with both A and B , *i.e.* $[A, [A, B]] = [B, [A, B]] = 0$, show that

$$e^A e^B = e^{A+B} e^{\frac{1}{2}[A, B]}. \quad (1)$$

[Hint: You might introduce a parameter α into the exponents by multiplying A and B by α . Show that Eq. (1) can be rewritten $e^{-\alpha A} e^{\alpha(A+B)} e^{-\alpha B} = e^{-\frac{1}{2}\alpha^2[A, B]}$ and derive a differential equation in the variable α obeyed by the left-hand side of this equation which you can solve.]