Beyond Unbounded Beliefs:
How Preferences and Information Interplay in Social Learning*

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Abstract

When does society eventually learn the truth, or take the correct action, via observational learning? In a general model of sequential learning over social networks, we identify a simple condition for learning dubbed *excludability*. Excludability is a joint property of agents' preferences and their information. When required to hold for all preferences, it is equivalent to information having "unbounded beliefs", which demands that any agent can individually identify the truth, even if only with small probability. But unbounded beliefs may be untenable with more than two states: e.g., it is incompatible with the monotone likelihood ratio property. Excludability reveals that what is crucial for learning, instead, is that a single agent must be able to displace any wrong action, even if she cannot take the correct action. We develop two classes of preferences and information that jointly satisfy excludability: (i) for a one-dimensional state, preferences with single-crossing differences and a new informational condition, directionally unbounded beliefs; and (ii) for a multi-dimensional state, Euclidean preferences and subexponential location-shift information.

Keywords: social learning; herds; information cascades; single crossing; Euclidean preferences; location-shift information; unbounded beliefs.

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1. Introduction

This paper concerns the classic sequential observational or social learning model initiated by Banerjee (1992) and Bikhchandani, Hirshleifer, and Welch (1992). There is an unknown payoff-relevant state (e.g., product quality). Each of many agents has homogeneous preferences over her own action and the state (e.g., all prefer products of higher quality). But agents act in sequence, each receiving her own private information about the state and observing some subset of her predecessors' actions. The central economic question is about asymptotic learning: do Bayesian agents eventually learn to take the correct action (e.g., will the highest quality product eventually prevail)?

One would anticipate that whether there is social learning depends on the combination of agents' preferences and their information structure. But, at least for finite action sets, economists have largely emphasized the latter dimension alone.[1] The reason is inextricably tied to focusing on models with two states. With only two states, there is social learning given any (nontrivial) preferences if and only if there is learning for all preferences. For, with two states, even the former requires private signals/beliefs to be *unbounded* (Smith and Sørensen, 2000; Acemoglu, Dahleh, Lobel, and Ozdaglar, 2011). Unbounded beliefs says that given any full-support prior it should be possible for a single private signal, however unlikely it is, to make an agent arbitrarily close to certain about the true state.

With multiple—i.e., more than two—states, unbounded beliefs still characterizes learning for all preferences (Arieli and Mueller-Frank, 2021).[2] However, it is now a very demanding condition. Consider, for instance, the canonical example of *normal information*: the state is $\omega \in \Omega \subset \mathbb{R}$ and agents' signals are drawn independently from a normal distribution with mean ω and fixed variance. With only two states, there is unbounded beliefs because a very high signal makes one arbitrarily convinced of the high state, while a very low signal makes one arbitrarily convinced of the low state. But with multiple states, normal information fails unbounded beliefs: given any full-support prior, there is an upper bound on how certain one can become about any non-extremal state based on observing one signal.[3] Is social learning doomed with multiple states for familiar information struc-

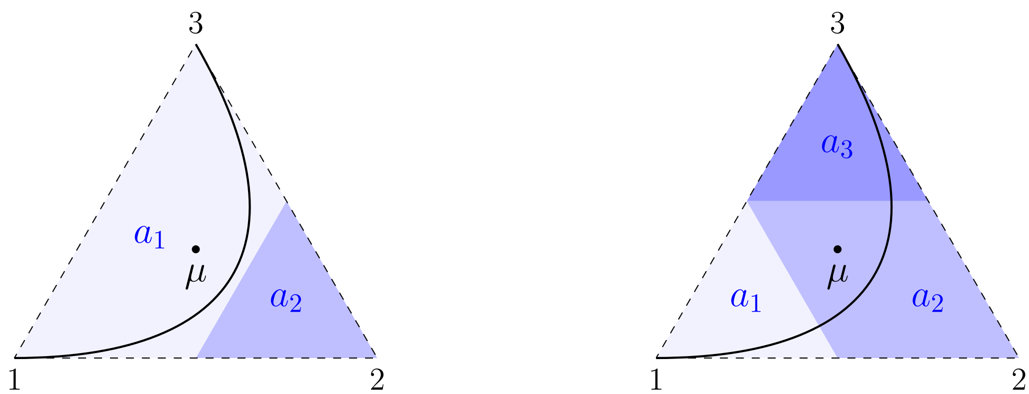
[1] Unless noted otherwise, our introduction should be understood as referring to the canonical sequential social learning model with a finite action set, homogeneous preferences, and no direct payoff externalities. It is well understood that variations in those aspects can also matter for social learning; see for example, Lee (1993) on infinite action spaces, Avery and Zemsky (1998) and Eyster, Galeotti, Kartik, and Rabin (2014) on endogenous prices or congestion costs, and Goeree, Palfrey, and Rogers (2006) on heterogeneous preferences.

[2] Arieli and Mueller-Frank (2021, Theorem 1) refer to the condition as "totally unbounded beliefs". They establish their result for a complete network, i.e., when each agent observes the actions of all predecessors. A by-product of our analysis is to establish it for general networks (Corollary 1 in Section 3).

[3] So binary states is special because all states are extreme states. There is nothing exceptional about normal

tures like normal information?

Our paper shows that the answer is no. With multiple states, whether there is learning depends on the interplay of preferences and information. Crucially, learning can obtain under standard preferences with familiar information structures that fail unbounded beliefs. Figure 1 illustrates an example of normal information with three states ($\Omega = \{1, 2, 3\}$), three actions ($A = \{a_1, a_2, a_3\}$), and a uniform prior. The failure of unbounded beliefs is reflected in the set of posteriors being bounded away from state 2's vertex. For concreteness, suppose that each agent observes all predecessors' actions. In Figure 1a, preferences are such that a_1 is optimal in both states 1 and 3, whereas a_2 is optimal in state 2; a_3 is dominated and thus ignored. Here, learning fails: since action a_1 is optimal no matter a single agent's signal, society is stuck with all agents taking a_1. By contrast, in Figure 1b, agents have standard quadratic-loss preferences: an agent who takes action a_i gets utility $-(i-\omega)^2$. Now, at any non-degenerate belief, no single action is optimal after all signals. This property yields social learning; see Theorem 1 in Section 3.



(a) $u(a_1, \omega) = -\frac{1}{2}$, $u(a_2, \omega) = -(2-\omega)^2$, $u(a_3, \omega) = -2$

(b) $u(a_i, \omega) = -(i-\omega)^2$

Figure 1: Belief simplex for state space $\Omega = \{1, 2, 3\}$. The curve depicts the set of posteriors for normal information given prior μ. Action set $A = \{a_1, a_2, a_3\}$ and each agent's utility is $u(a, \omega)$. The shaded regions depict optimal actions under uncertainty.

Excludability. Our paper develops a simple joint condition on information and preferences, which we call *excludability*, that is not only sufficient for social learning on general observational networks (satisfying a mild condition known as expanding observations), but in a sense also necessary; see Theorem 2 in Section 3.

Roughly speaking, excludability requires that for each pair of actions, a and a', a single agent must be able to receive a signal that makes her arbitrarily convinced that a is better

information violating unbounded beliefs; see Remark 2 in Section 3.

than a', no matter which (full-support) belief she starts with. Put differently, information must be able to distinguish the set of states in which a is better than a' from the set in which a' is better than a. Excludability implies that society can never get stuck on a wrong action: if an action is suboptimal at the true state, then some agent will receive a private signal convincing her not to take that action. We establish that this property of *displacing wrong actions* leads to social learning. Notably, an agent can displace wrong actions even if she cannot take the correct action, i.e., the optimal action at the true state. (See Figure 2 in Section 3 for a concrete example.) By contrast, under unbounded beliefs, it is always possible for a single agent's private signal to induce her to take the correct action. We view the distinction of social learning arising from the individual capacity to displace wrong actions rather than to discover the correct action as an insight of this paper; this distinction cannot be seen with only two states.

Excludability provides a useful perspective on existing ideas in the literature. For instance, as detailed in Section 3, an information structure yields excludability for all preferences if and only if that information structure has unbounded beliefs. But more importantly, we can use excludability to deduce tenable conditions on information—weaker than unbounded beliefs—that are sufficient for social learning in general observational networks for canonical classes of preferences. This approach to relaxing stringent conditions is classical in other areas of economics,[4] but is novel to social learning.

Single-crossing preferences. Our leading application of excludability is to preferences with *single-crossing differences* (SCD). Here we show that learning obtains when the information structure satisfies *directionally unbounded beliefs* (DUB). SCD is a familiar property (Milgrom and Shannon, 1994) that is widely assumed in economics: it captures settings in which there are no preference reversals as the state increases. By contrast, DUB appears to be a new condition on information structures, although Milgrom (1979) utilizes a related property in the context of auction theory. Like SCD, DUB is formulated for a (totally) ordered state space. It requires that for any state ω and any prior that puts positive probability on ω, there exist both: (i) signals that make one arbitrarily certain that the state is at least ω; and (ii) signals that make one arbitrarily certain that the state is at most ω. Crucially, no signal need make one arbitrarily certain about ω (unlike unbounded beliefs). For the normal information structure discussed earlier, requirements (i) and (ii) are met for any state by arbitrarily high and arbitrarily low signals, respectively.

[4] For instance, first-order stochastic dominance is refined to second-order stochastic dominance by restricting to concave (and increasing) utility functions. Another exemplar is refining the Blackwell (1953) "sufficiency" partial order on information structures to "accuracy" by restricting preferences to satisfy single crossing or interval dominance (Lehmann, 1988; Persico, 2000; Quah and Strulovici, 2009).

Proposition 1 in Section 4 shows that SCD preferences and DUB information are jointly sufficient for excludability, and hence learning. For a direct intuition on the SCD-DUB interplay, consider normal information again. There are preferences (like those in Figure 1a) under which society can get stuck at some belief at which agents are taking an incorrect action, but only a strong signal about an intermediate state would change the action—alas no such signal is available. However, under SCD preferences (like those in Figure 1b), if knowing that the state is some intermediate ω would change the action, then so would knowing that the state is at least ω or at most ω. Normal information, or more generally DUB, guarantees that there are strong signals approximating such knowledge.

Euclidean preferences. Our second application in Section 4 is to *weighted Euclidean preferences* in multidimensional spaces, where the state is $\omega \in \mathbb{R}^d$ and the action is $a \in \mathbb{R}^d$. These subsume the canonical Euclidean preferences invoked in political economy and communication/delegation models, $u(a, \omega) = -\sum_{i=1}^{d}(a_i - \omega_i)^2$.

Using excludability, we show that social learning obtains under weighted Euclidean preferences so long as information is given by a *subexponential location-shift family*. Location-shift families are widely-used information structures: for some density $g : \mathbb{R}^d \to \mathbb{R}^d$, the signal distribution in any state ω is given by $g(s - \omega)$. Loosely, our subexponential condition requires that the tail of g must be thin enough, eventually decreasing faster than an exponential rate. We establish that this thin-tails property guarantees excludability under weighted Euclidean preferences. Notably, multidimensional normal information (i.e., normally distributed signals with mean equal to the state and some fixed covariance matrix) satisfies the subexponential requirement.

Methodology. A significant contribution of our paper is also methodological. We develop an approach to tackle learning, and more generally, asymptotic social welfare with multiple states in general observational networks. Theorem 1 in Section 3 is the backbone by which we tie learning to excludability. Theorem 1 reduces the complex dynamic problem of social learning in networks to a much simpler "static" problem. The theorem says that there is learning if and only if every *stationary belief* has *adequate knowledge*. A stationary belief is one at which there is an action that is optimal no matter the signal, and an adequate-knowledge belief is one at which there is an action that is optimal no matter the state in the belief's support. Excludability is a simple sufficient condition for all stationary beliefs to have adequate knowledge. Moreover, subject to a technical qualifier, it is also necessary if society may have to choose from an arbitrary subset of actions.

Theorem 1 itself is a consequence of Theorem 3 in Section 5, which is a welfare result that applies even when learning fails. This theorem establishes a welfare lower bound: roughly, for any preferences and information (and given expanding observations), agents eventually obtain at least their *cascade utility*. Cascade utility is the minimum expected utility an agent can get from any Bayes-plausible distribution of stationary beliefs. Theorem 3 implies that learning obtains when the cascade utility equals the utility obtained from taking the correct action in each state, which leads to Theorem 1.

Related literature. A number of papers on sequential Bayesian social learning only consider the complete observational network: each agent observes all her predecessors' actions. For that case and with binary states, Smith and Sørensen (2000) show that, given any nontrivial preferences, there is learning if and only if beliefs are unbounded. For the complete network but with multiple states, Arieli and Mueller-Frank (2021, Theorem 1) show that unbounded beliefs—which they call "totally unbounded beliefs"—is sufficient for learning, and also necessary if learning must obtain no matter society's preferences.[5] The approach of both Smith and Sørensen (2000) and Arieli and Mueller-Frank (2021) rests on the social belief—an agent's belief based on observing her predecessors' actions, before observing her own signal—being a martingale in the complete network.

Gale and Kariv (2003) and Çelen and Kariv (2004) depart from the complete network, noting that martingale methods now fail. Both these papers also depart from the canonical setting in other ways, however: in Gale and Kariv (2003) agents choose actions repeatedly, while in Çelen and Kariv (2004) private signals are not independent conditional on the true state. Acemoglu, Dahleh, Lobel, and Ozdaglar (2011) provide a general treatment of observational networks in an otherwise classical setting. But they only allow for binary states and binary actions. They introduce the condition of expanding observations, explaining that this property of the network is necessary for learning. Their Theorem 2 establishes that it is also sufficient for learning with unbounded beliefs. Building on Banerjee and Fudenberg (2004), a key contribution of Acemoglu, Dahleh, Lobel, and Ozdaglar (2011) is to use a welfare *improvement principle* to deduce learning; this approach works even though martingale arguments fail. Lobel and Sadler (2015) introduce a notion of "information diffusion" and use the improvement principle to establish information diffusion even when learning fails.

The analysis in both Acemoglu, Dahleh, Lobel, and Ozdaglar (2011) and Lobel and

[5] The early work of Bikhchandani, Hirshleifer, and Welch (1992) allowed for multiple states, but they only identified failures of learning because they implicitly restricted attention to bounded beliefs; more precisely, they assumed finite signals with full-support distributions.

Sadler (2015) relies on their binary-state binary-action structure.[6] We believe ours is the first paper to consider the canonical sequential social learning problem with general observational networks and general state and action spaces. At a methodological level, we develop a novel analysis based on continuity and compactness—rather than monotonicity or other properties that are specific to binary states or actions—that uncovers the fundamental logic underlying a general improvement principle.

Substantively, our focus on multiple states and actions allows us to shed light on how preferences and information jointly shape social learning. As already noted, their interplay in determining learning has not received attention in the prior literature because of its focus on binary states. The only exception we are aware of is Arieli and Mueller-Frank (2021, Theorem 3), discussed in Section 3; their result assumes a special utility function and is only for the complete network.

2. Model

There is a countable (finite or infinite) state space Ω, endowed with the discrete topology, and standard Borel spaces of actions A and signals S. An information or signal structure is given by a collection of probability measures over S, one for each state, denoted by $F(\cdot|\omega)$. Assume that for any ω and ω', $F(\cdot|\omega)$ and $F(\cdot|\omega')$ are mutually absolutely continuous. It follows that each $F(\cdot|\omega)$ has a density $f(\cdot|\omega)$; more precisely, this is the Radon-Nikodym derivative of $F(\cdot|\omega)$ with respect some reference measure that is mutually absolutely continuous with every $F(\cdot|\omega')$. Without further loss of generality we assume $f(\cdot|\cdot) > 0$, so that no signal excludes any state.

The game. At the outset, a state ω is drawn from a common prior probability mass function $\mu_0 \in \Delta\Omega$.[7] Then, an infinite sequence of agents, indexed by $n = 1, 2, \ldots$, sequentially select actions. An agent n observes both a private signal s_n drawn from $f(\cdot|\omega)$ and the actions of some subset of her predecessors $B_n \subseteq \{1, 2, \ldots, n-1\}$, and then chooses her action $a_n \in A$. Agents' private signals are drawn independently conditional on the state, and no agent observes either the state or any of her predecessors' signals. Each observational neighborhood B_n is stochastically generated according to a probability distribution Q_n over all subsets of $\{1, 2, \ldots, n-1\}$, assumed to be independent across n, independent of the state ω, and independent of any private signals. The distributions $(Q_n)_{n\in\mathbb{N}}$ constitute

[6] Banerjee and Fudenberg (2004) and Smith and Sørensen (2020) consider "unordered" random sampling models that also only allow for binary states and actions.

[7] For any topological space X, ΔX denotes the set of Borel probability measures over X.

the observational network structure and are common knowledge, but the realized neighborhood B_n is the private information of agent n.

Agent n's information set thus consists of her signal s_n, neighborhood B_n, and the actions chosen by the neighbors $(a_k)_{k \in B_n}$.[8] Let $\mathcal{I}_n$ denote the set of all possible information sets for agent n. A strategy for agent n is a (measurable) function $\sigma_n : \mathcal{I}_n \to \Delta A$.

All agents are expected utility maximizers and have common preferences that depend only on their own action and the state, represented by the utility function $u : A \times \Omega \to \mathbb{R}$. We assume that utility is bounded: there is $\overline{u} \geq 0$ such that $|u(\cdot, \cdot)| \leq \overline{u}$.

We study the Bayes Nash equilibria—hereafter simply equilibria—of this game. We assume that for every belief there is an optimal action, so that an equilibrium exists.[9]

Remark 1. Appendix A describes a more general setting in which our main results are proved. For example, Ω can be a closed subset of $\mathbb{R}$ and each $u(a, \cdot)$ piecewise continuous with A finite.

Adequate learning. The full-information expected utility given a belief μ is the expected utility under that belief if the state will be revealed before an action is chosen:

$$u^*(\mu) := \sum_{\omega \in \Omega} \max_{a \in A} u(a, \omega) \mu(\omega).$$

Given a prior μ_0 and a strategy profile σ, agent n's utility u_n is a random variable. Let $\mathbb{E}_{\sigma,\mu_0}[u_n]$ be agent n's ex-ante expected utility. We say there is *adequate learning* if for every prior μ_0 and every equilibrium σ, $\mathbb{E}_{\sigma,\mu_0}[u_n] \to u^*(\mu_0)$. In words, adequate learning requires that given any prior and equilibrium, no matter which state is realized, eventually agents take actions that are arbitrarily close to optimal in that state.[10] We say there is *inadequate learning* if adequate learning fails.[11]

[8] While we assume that each agent observes the identities of her neighbors as well as their chosen actions, the Conclusion explains how our analysis extends to various cases of "random sampling" in which neighbors' identities are not observed. Our analysis also applies if agents receive arbitrary information about their predecessors' realized neighborhoods.

[9] Existence of optimal actions is assured under standard assumptions, e.g., if A is compact and $u(\cdot, \cdot)$ is suitably continuous. We also note that as there are no direct payoff externalities, strategic interaction is minimal: any σ_n affects other agents only insofar as affecting how n's successors update about signal s_n from the observation of action a_n. Hence, we could just as well adopt (weak) Perfect Bayesian equilibrium or refinements.

[10] Our notion of adequate learning is different from Arieli and Mueller-Frank's (2021), who require learning for all utility functions. Following Aghion, Bolton, Harris, and Jullien (1991), we use "adequate" to signify that learning the state precisely is not necessary when some action is optimal in multiple states.

[11] That we deem learning to be inadequate if there is some equilibrium in which learning fails, rather than in every equilibrium, is innocuous given that there is no strategic interaction (cf. fn. 9). On the other hand,

We will also be interested in situations in which agents have to choose from a subset of actions, referred to as a *choice set*.[12] We say that there is *(in)adequate learning for a choice set* $\tilde{A} \subseteq A$ if there is (in)adequate learning when agents are restricted to choose from actions in $\tilde{A}$.

Expanding observations. As observed by Acemoglu, Dahleh, Lobel, and Ozdaglar (2011), a necessary condition for adequate learning is that the network structure has *expanding observations*:

$$\forall K \in \mathbb{N} : \lim_{n\to\infty} Q_n (B_n \subseteq \{1, \ldots, K\}) = 0. \tag{1}$$

The reason is that a failure of expanding observations means that for some $K \in \mathbb{N}$, there is an infinite number of agents each of whom, with probability uniformly bounded away from 0, observes the actions of only at most the first K agents and hence cannot do better than choosing their action based on only $K + 1$ signals.

Accordingly, we assume expanding observations. Leading examples of network structures with expanding observations include: (i) the classic complete network in which each agent's neighborhood is all her predecessors (formally, $Q_n(B_n = \{1, \ldots, n-1\}) = 1$); (ii) each agent only observes her immediate predecessor ($Q_n(B_n = \{n-1\}) = 1$); and (iii) each agent observes a random predecessor ($Q_n(B_n = \{k\}) = 1/(n-1)$ for all $k \in \{1, \ldots, n-1\}$).

3. Characterizations of Learning

3.1. Stationary Beliefs and Adequate Knowledge

The backbone for our analysis is Theorem 1 below, which simplifies the question of adequate learning to a "one-shot updating" property of beliefs. To state that result, we require two concepts concerning the value of information.

For any belief $\mu \in \Delta\Omega$, let $c(\mu) := \arg\max_{a\in A} \mathbb{E}_\mu[u(a, \omega)]$ denote the set of optimal actions under that belief. Abusing notation, for a degenerate belief on state ω we write $c(\omega)$. Denoting the posterior after signal s when starting from belief μ by μ_s, we say that belief μ is *stationary* if there is $a \in c(\mu)$ such that $a \in c(\mu_s)$ for μ-a.e. signals s. We say that belief μ has *adequate knowledge* if there is $a \in c(\mu)$ such that $a \in c(\omega)$ for all $\omega \in \operatorname{Supp} \mu$. So a belief is stationary if an agent holding that belief does not benefit from observing a signal from the

the issue of whether learning fails at every prior rather than only at some priors is substantive. We return to this issue in our Conclusion.

[12] We restrict attention to choice sets such that for every belief there is an optimal action.

given information structure; whereas a belief has adequate knowledge if the agent would not benefit from observing a signal from *any* information structure.

Theorem 1. *There is adequate learning if and only if all stationary beliefs have adequate knowledge.*

Theorem 1 provides a characterization of adequate learning that holds regardless of the observational network structure, given our maintained assumption of expanding observations. Its "only if" direction is straightforward because our notion of learning considers all priors: if the prior is stationary and has inadequate knowledge, then society is stuck with all agents taking the prior-optimal action even though it is suboptimal in some states. More important and subtle is the theorem's "if" direction. It is inspired by earlier results, particularly Arieli and Mueller-Frank (2021, Lemma 1) and Lobel and Sadler (2015, Theorem 1), but the logic in the current general setting of arbitrary networks and multiple states and actions is novel. We defer this logic to Section 5, instead turning now to how we build on Theorem 1 for a more practicable characterization of learning.

3.2. Excludability

Since stationary beliefs are not a primitive of the model, Theorem 1 is not always transparent about whether any given combination of preferences and information lead to adequate learning. We thus turn to identifying a simple condition on primitives for adequate learning.

A key notion is whether information allows an agent to become arbitrarily sure about a subset of states Ω' relative to another subset Ω''. Formally, writing $\mu_s(\Omega')$ for the posterior on states Ω' induced by belief μ and signal s, and $\Pr_\mu(S')$ for the probability of signal set S' induced by belief μ:

Definition 1. A set Ω' is *distinguishable* from another set Ω'' if for any $\varepsilon > 0$ and $\mu \in \Delta(\Omega' \cup \Omega'')$ with $\mu(\Omega') > 0$, it holds that $\Pr_\mu(s : \mu_s(\Omega') > 1 - \varepsilon) > 0$.

Note that Ω' is distinguishable from Ω'' if and only if every $\omega \in \Omega'$ is distinguishable from Ω''. Moreover, if Ω' is distinguishable from Ω'', then every subset of Ω' is distinguishable from every subset of Ω''. The following observation essentially reinterprets distinguishability directly in terms of the signal structure rather than posteriors.

Lemma 1. *Ω' is distinguishable from Ω'' if for every $\omega' \in \Omega'$ and $\varepsilon > 0$, there is a positive-probability set of signals S' such that*

$$\forall \omega'' \in \Omega'', \forall s \in S' : \frac{f(s|\omega'')}{f(s|\omega')} < \varepsilon.$$

Conversely, this condition is also necessary if Ω'' is finite.

It bears emphasis that the set S' in the lemma cannot depend on $\omega'' \in \Omega''$; for Ω' to be distinguished from Ω'', each $\omega' \in \Omega'$ must be distinguished from all $\omega'' \in \Omega''$ simultaneously. Consider the example of *normal information*: $\Omega \subset \mathbb{R}$ and signals are normally distributed on $\mathbb{R}$ with mean ω and fixed variance. When $\Omega = \{1, 2, 3\}$, state 2 is distinguishable from 1 because $f(s|1)/f(s|2) \to 0$ as $s \to \infty$, and state 2 is distinguishable from 3 because $f(s|3)/f(s|2) \to 0$ as $s \to -\infty$. But state 2 cannot be distinguished from both 1 and 3 simultaneously, because $\min\{f(s|1)/f(s|2), f(s|3)/f(s|2)\}$ is bounded away from 0.

Distinguishability of each state from its complement is the condition of *unbounded beliefs*; this is termed "totally unbounded beliefs" by Arieli and Mueller-Frank (2021) and is the multi-state extension of the two-state notion introduced by Smith and Sørensen (2000). But unbounded beliefs is highly restrictive with more than two states; for instance, with ordered states and signals, it is incompatible with the ubiquitous monotone likelihood ratio property (MLRP) in information economics:[13]

Remark 2. Under any MLRP information structure, no state ω is distinguishable from any $\{\omega', \omega''\}$ if $\omega' < \omega < \omega''$. Consequently, if $|\Omega| > 2$ and the MLRP holds, then unbounded beliefs fails.

Fortunately, learning only requires certain subsets of states to be distinguished from each other. Given utility function $u(a, \omega)$ and any two actions a_1 and a_2, let the preferred set $\Omega_{a_1,a_2} := \{\omega : u(a_1, \omega) > u(a_2, \omega)\}$ be the set of states in which a_1 is strictly preferred to a_2.

Definition 2. A utility function and an information structure jointly satisfy *excludability* if for every pair of actions a_1 and a_2, Ω_{a_1,a_2} is distinguishable from Ω_{a_2,a_1}.

Excludability is a simple primitive condition on preferences and information. It requires that for any pair of actions, a single agent can become arbitrarily certain that one action is strictly better than the other, starting from any belief that does not exclude that event. Notably, for any given preferences, Lemma 1 reduces excludability to a set of likelihood-ratio conditions on the information structure, without reference to beliefs.

An information structure yields excludability for all preferences if and only if it has unbounded beliefs: sufficiency is immediate; for necessity, note that if ω is not distinguishable from its complement, then excludability fails when preferences are such that for some a_1

[13] For ordered state and signals spaces, the MLRP holds if $\forall s' > s$ and $\forall \omega' > \omega$, $f(s|\omega')/f(s|\omega) \leq f(s'|\omega')/f(s'|\omega)$.

and a_2, $\Omega_{a_1,a_2} = \{\omega\}$ while $\Omega_{a_2,a_1} = \Omega \setminus \{\omega\}$.[14] In fact, with only two states, excludability under any given nontrivial preferences is equivalent to unbounded beliefs. But with multiple states, interesting combinations of preferences and information yield excludability even absent unbounded beliefs, as elaborated in Section 4. This matters because:

Theorem 2. *Excludability implies adequate learning for every choice set. If excludability fails and the number of states is finite, then there is inadequate learning for some choice set.*

Excludability is sufficient for adequate learning because it ensures that wrong actions can always be "displaced", which by Theorem 1 is the key to social learning. More precisely, excludability guarantees that, at every choice set, all stationary beliefs have adequate knowledge. Suppose a belief μ has *in*adequate knowledge, so that $c(\mu) \neq c(\omega^*)$ for some state $\omega^* \in \operatorname{Supp} \mu$. (For simplicity, assume $c(\mu)$ and $c(\omega^*)$ are singletons.) Excludability implies that state ω^* is distinguishable from the set of states $\Omega_{c(\mu),c(\omega^*)}$. Hence, with positive probability, an agent who starts with belief μ will obtain a posterior that puts arbitrarily large probability on ω^* relative to $\Omega_{c(\mu),c(\omega^*)}$, in which event she strictly prefers $c(\omega^*)$ to $c(\mu)$. Consequently, μ is not stationary.

It bears noting, however, that excludability does not guarantee that a wrong action can always be displaced by the correct action. In other words, even though excludability guarantees that given any wrong action—say, $c(\mu)$ when the true state is ω^*—a single agent can receive a signal convincing her that $c(\mu)$ is worse than the correct action $c(\omega^*)$, there may be no signal that leads the agent to take $c(\omega^*)$. When there are two states and finite actions, always being able to displace a wrong action and always being able to take the correct action are equivalent, as they both reduce to unbounded beliefs. But more generally, it is displacing wrong actions that is fundamental for learning.

To illustrate the point concretely, consider the example depicted in Figure 2. There are three states and three actions, $\Omega = A = \{1, 2, 3\}$. The signal structure and preferences are detailed in the figure's caption. The correct action in each state ω is $a = \omega$. Importantly, unbounded beliefs fails yet there is excludability.[15] Let the *social belief* at time n be agent n's belief about the state given only the history of her neighbors' actions, prior to observing her own private signal. When each agent observes all predecessors' actions, Figure 2 shows

[14] On the other hand, if the information structure has bounded beliefs—no state is distinguishable from any other—then excludability fails under any given nontrivial preferences. (Preferences are nontrivial if there is no action that is optimal at all states.)

[15] Unbounded beliefs fails because under normal information the middle state is not distinguishable from its complement. Excludability can be verified by checking distinguishability of the preferred sets for each pair of actions; alternatively, we note that the preferences satisfy single-crossing differences (SCD), and as explained in Subsection 4.1, SCD and normal information imply excludability.

two numerically-simulated paths of social beliefs given the true state $\omega = 2$. The social belief starts at the prior, marked by a star in the figure, and then evolves as agents take actions, as indicated by either of the arrowed paths. Notice that there is a range of beliefs, shaded in grey, such that for any social belief in that range no signal can lead an agent to take the correct action 2. As the prior is in this range, *the first agent necessarily takes a wrong action*: either 1 (which occurs in the red path) or 3 (the blue path). Nevertheless, even though no agent can take the correct action 2 for a while, society never gets stuck at a wrong action: given that an agent's predecessor chose $a \in \{1, 3\}$, there are signals (very high if $a = 1$ and very low if $a = 3$) that convince the agent that a is worse than the correct action 2, and hence the agent will not take action a. At some point, after enough switching between actions 1 and 3, the social belief is driven outside the grey region and it becomes possible for an agent to take the correct action 2. Eventually, society settles on that action.

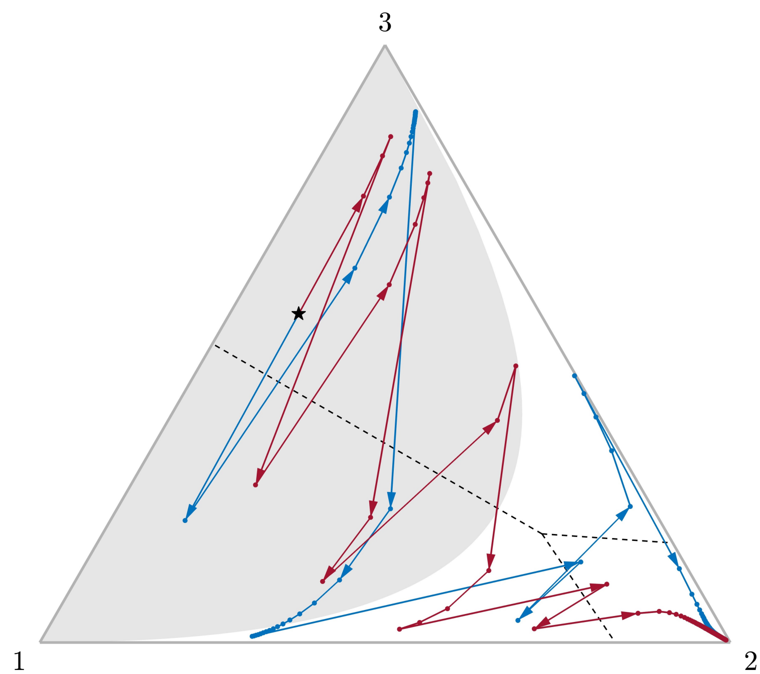


Figure 2: Two simulated social belief paths—one in red and one in blue—in a complete network. There are three states labeled $1, 2, 3$, and there is normal information (with standard deviation 1.2). There are three actions with respective state-contingent utilities $(1, 0, -0.3)$, $(0, 0.2, 0)$, and $(-0.3, 0, 1)$. The optimal action under uncertainty is delineated by the dashed lines. The true state is 2, and society starts with the prior $(0.35, 0.1, 0.55)$, marked by the black star. The grey shaded region indicates beliefs at which no signal can lead to state 2's correct action. On each path, a dot represents the social belief after an agent has acted, and arrows indicate the sequencing.

Turning to necessity in Theorem 2: for a fixed choice set, all stationary beliefs can have adequate knowledge (and hence there is adequate learning, by Theorem 1) even absent excludability. But when excludability fails, there is some preferred set Ω_{a_1,a_2} that cannot be distinguished from Ω_{a_2,a_1}. If Ω is finite, this means that when the choice set is $\tilde{A} = \{a_1, a_2\}$, a belief that puts small probability on Ω_{a_1,a_2} relative to Ω_{a_2,a_1} is stationary and has inadequate knowledge.[16] Hence, Theorem 1 implies that excludability is necessary for learning when we seek learning for all choice sets. The following example illustrates.

Example 1. Consider $\Omega = \{0, 1\}$, $A = [0, 1]$, and $u(a, \omega) = -(a - \omega)^2$. This is an example of "responsive preferences" (Lee, 1993; Ali, 2018). Fix any nontrivial signal structure and any observational network structure satisfying expanding observations.

Adequate learning obtains by Theorem 1, because the only stationary beliefs have certainty on one of the two states. For, given any nondegenerate belief, with positive probability the posterior-optimal action will be different from the prior-optimal action, as the uniquely optimal action equals the posterior expected state. However, excludability is equivalent to the signal structure having unbounded beliefs, as for any $a_1 < a_2$, $\Omega_{a_1,a_2} = \{0\}$ and $\Omega_{a_2,a_1} = \{1\}$. So excludability is not necessary for adequate learning at the choice set A. But absent excludability there is *in*adequate learning at any non-singleton finite choice set. For, there is then some state such that any prior that puts probability close to 1 on that state will be stationary, but this prior has inadequate knowledge. $\square$

The choice-set variation required by Theorem 2 comes "for free" when we seek an informational condition that ensures learning for a broad-enough class of preferences. Specifically, the class must be such that for any pair of actions, there is a preference in the class such that only these two actions are undominated. Since excludability for all preferences is equivalent to the information structure having unbounded beliefs, Theorem 2 immediately implies:

Corollary 1. *An information structure yields adequate learning for all preferences if and only if it has unbounded beliefs.*

This corollary subsumes results from the prior literature, which are either for the complete network (Arieli and Mueller-Frank, 2021, Theorem 1) or general networks but with only two states (Acemoglu, Dahleh, Lobel, and Ozdaglar, 2011, Theorem 2).

To our knowledge, the only exception to unbounded beliefs driving learning with a discrete action space is the interesting example of Arieli and Mueller-Frank (2021, Theorem 3). They consider the complete network and a special utility function, which they call "simple utility", in which the payoff is 1 if the action matches the state and 0 otherwise.

[16] The qualifier that Ω is finite is technical; it ensures that $\min_{\omega \in \Omega_{a_1,a_2}} [u(a_1, \omega) - u(a_2, \omega)]$ exists. Theorem 2′ in the appendix is a more general version of Theorem 2 that does not require finiteness.

For this case, they show that *pairwise distinguishability*—for any pair of states, each is distinguishable from the other—is sufficient for learning. We note that this too follows from Theorem 2; indeed, the theorem implies that learning obtains for general observational networks. For, under simple utility, given any two actions $a_1 \neq a_2$, the preferred sets Ω_{a_1,a_2} and Ω_{a_2,a_1} are just $\{a_1\}$ and $\{a_2\}$. Hence, under simple utility, an information structure yields excludability if and only if it has pairwise distinguishability.

More generally, though, the true value of excludability is that it permits a study of broad classes of preferences and information that are mutually sufficient for learning. We proceed to applications.

4. Applications

This section identifies informational conditions that guarantee adequate learning for two widely-used classes of preferences: with a one-dimensional state, preferences with single-crossing differences; and with a multi-dimensional state, (weighted) Euclidean preferences. These applications showcase how excludability can be fruitfully decoupled into separate conditions on preferences and information.

4.1. Learning with Single-Crossing Preferences

In this subsection we take the state space to be totally ordered: for simplicity, $\Omega \subset \mathbb{R}$. A function $h : \Omega \to \mathbb{R}$ is *single crossing* if either: (i) for all $\omega < \omega'$, $h(\omega) > 0 \implies h(\omega') \geq 0$; or (ii) for all $\omega < \omega'$, $h(\omega) < 0 \implies h(\omega') \leq 0$. That is, a single-crossing function switches sign between strictly positive and strictly negative at most once.

Definition 3. Preferences represented by $u : A \times \Omega \to \mathbb{R}$ have *single-crossing differences* (SCD) if for all a and a', the difference $u(a, \cdot) - u(a', \cdot)$ is single crossing.

SCD is an ordinal property closely related to notions in Milgrom and Shannon (1994) and Athey (2001), but, following Kartik, Lee, and Rappoport (2023), the formulation is without an order on A.[17] Ignoring indifferences, SCD requires that the preference over any pair of actions can only flip once as the state changes monotonically. SCD is widely satisfied in economic models; in particular, it is assured by supermodularity of u.

The key informational condition is that of distinguishing upper and lower sets from each other. More precisely, we require that for any ω, $\{\omega' : \omega' \geq \omega\}$ and $\{\omega' : \omega' < \omega\}$ are

[17] SCD is equivalent to there existing some order on A with respect to which Athey's (2001) "weak single-crossing property of incremental returns" holds.

distinguishable from each other, and $\{\omega' : \omega' > \omega\}$ and $\{\omega' : \omega' \leq \omega\}$ are distinguishable from each other. But since a set Ω' is distinguishable from Ω'' if and only if each $\omega \in \Omega'$ is distinguishable from Ω'', we can simplify as follows:

Definition 4. An information structure has *directionally unbounded beliefs* (DUB) if every ω is distinguishable from $\{\omega' : \omega' < \omega\}$ and also from $\{\omega' : \omega' > \omega\}$.

Crucially, DUB does not require that any state ω is distinguishable from any subset of states containing both some higher and some lower states than ω. That would be incompatible with the MLRP when there are multiple states (recall Remark 2). Rather, using Lemma 1, we can view DUB as only requiring that for any state ω, there are signals that are arbitrarily more likely in ω relative to all $\omega' < \omega$, and also other signals that are arbitrarily more likely in ω relative to all $\omega' > \omega$.[18]

A leading example of DUB information is normal information. Indeed, normal information has the MLRP and pairwise distinguishability (e.g., arbitrarily high signals distinguish any state from any lower state), which together imply DUB.[19] Absent the MLRP, pairwise distinguishability can hold without DUB; this is the case, for instance, in Example 1 of Arieli and Mueller-Frank (2021). But even without the MLRP, DUB holds for any subexponential location-shift information structure, as elaborated in Subsection 4.2.

We can now state:

Proposition 1. *If preferences have SCD and the information structure has DUB, then there is adequate learning. Conversely, if the information structure violates DUB and there are at least two actions, then there are SCD preferences for which there is inadequate learning.*

The result says that not only is DUB a sufficient informational condition for adequate learning under any SCD preferences, but it also necessary to assure learning for all SCD preferences.

Here is the logic for sufficiency. Recall that $\Omega_{a,a'}$ denotes the states in which action a is strictly preferred to a'. SCD implies non-reversal of strict preferences: for any a and a',

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[18] In the context of auctions, Milgrom (1979) is concerned with distinguishing each state only from its lower set, and he notes in his Theorem 2 the implication of our Lemma 1 for that distinguishability. As he mentions, it is possible that any/all states are distinguishable from their lower sets but not their upper sets, or vice-versa, or they can be distinguishable from both or neither.

[19] Regardless of the MLRP, DUB implies pairwise distinguishability. To see why the converse is true given the MLRP, consider the case of finite states. Note that for any $\omega' > \omega$, $f(s|\omega')/f(s|\omega) \to \infty$ as $s \to \sup S$ (the ratio is increasing by MLRP, and it diverges by pairwise distinguishability); similarly, the ratio goes to 0 as $s \to \inf S$. Hence, for any ω' and $\varepsilon > 0$, the condition in Lemma 1 is met for $\Omega' = \{\omega'\}$ and $\Omega'' = \{\omega'' : \omega'' < \omega'\}$ when S' is any sufficiently small upper set of signals, while for $\Omega'' = \{\omega'' : \omega'' > \omega'\}$ the condition is met when S' is any sufficiently small lower set. For an infinite state space, the intuition is the same but we must appeal to the monotone convergence theorem.

either $\inf \Omega_{a,a'} \geq \sup \Omega_{a',a}$ or $\inf \Omega_{a',a} \geq \sup \Omega_{a,a'}$. DUB is equivalent to saying that every upper (resp., lower) set of states and its strict lower (resp., strict upper) set are distinguishable from each other. Therefore, SCD and DUB together guarantee excludability, and so Proposition 1's first statement follows from Theorem 2.

The argument for necessity of DUB in Proposition 1 is as follows. Assume DUB fails: say, some state ω^* cannot be distinguished from its lower states (it is analogous if "lower" is replaced by "upper"). Define the following SCD utility: for all $\omega < \omega^*$, $u(a_1, \omega) = 1$ and $u(a_2, \omega) = 0$; for all $\omega \geq \omega^*$, $u(a_1, \omega) = 0$ and $u(a_2, \omega) = 1$. Any other actions can be assumed to be strictly dominated and ignored. The failure of DUB implies that if a prior μ_0 is supported on $\omega \leq \omega^*$ and $\mu_0(\omega^*) > 0$ is small enough, then a_2 is never chosen after any signal. Any such prior is stationary but has inadequate knowledge. By Theorem 1, there is inadequate learning.[20]

While our main point in this subsection is that DUB is the correct informational condition for adequate learning under SCD preferences, it is also worth noting that there can be inadequate learning under DUB information absent SCD preferences. The following example illustrates.

Example 2. Let $\Omega = \mathbb{Z}$ and $A = \mathbb{Z} \cup \{a^*\}$. In any state ω, the utility from any integer action a is given by quadratic loss, $u(a, \omega) = -(a - \omega)^2$, whereas the action a^* is a "safe action", $u(a^*, \omega) = -\varepsilon$ for a small constant $\varepsilon > 0$.[21] So any action ω is uniquely optimal in state ω but worse than the safe action a^* in every other state. Plainly, SCD is violated.

Consider normal information. There are full-support priors such that the posterior probability of any state is uniformly bounded away from 1 across all the signals and states (details are available in Kartik, Lee, Liu, and Rappoport, 2022). For any such prior, for small enough $\varepsilon > 0$, the safe action a^* is optimal after every signal. In other words, any such prior is stationary but has inadequate knowledge, and so Theorem 1 implies that there is inadequate learning. $\square$

Remark 3. If Ω is finite, then DUB information yields adequate learning if there are distinct optimal actions at all states, i.e., for any $\omega \neq \omega'$, $c(\omega) \cap c(\omega') = \emptyset$. This follows from Theorem 1, because the only stationary beliefs are degenerate: for any belief μ, DUB implies that the posterior can be arbitrarily certain on the extreme states of μ's support. But Example 2

[20] Put differently, this argument shows that when DUB fails, there are SCD preferences such that excludability fails; the failure of adequate learning then follows from Theorem 2, or more precisely its infinite-state counterpart in the appendix, Theorem 2′. Note that the choice-set variation required by Theorem 2 is assured because SCD preferences are rich enough to stipulate any action as strictly dominated.

[21] Strictly speaking, quadratic-loss utility with $\Omega = \mathbb{Z}$ violates our maintained assumption of bounded utility, but we ignore that to keep the example succinct.

cautions that pairing DUB with distinct optimal actions at all states is not a robust principle for social learning. Furthermore, for some DUB information structures such as normal information, excludability fails whenever SCD is violated; hence, Theorem 2 implies there is inadequate learning at some choice set. Indeed, because SCD and DUB are a minimal pair of sufficient conditions for excludability in the terminology of Athey (2002)—if one condition fails, then the other can be satisfied though the preferences-information pair jointly fails excludability—Theorem 2 implies that SCD and DUB are a minimal pair of sufficient conditions for adequate learning at all choice sets.

4.2. Learning with Euclidean Preferences

In this subsection, we turn to a multidimensional state space and take $A, \Omega \subset \mathbb{R}^d$ for some integer $d \geq 1$. We view any $x \in \mathbb{R}^d$ as a column vector and denote its transposition by x' and its standard Euclidean norm by $\|x\|$.

Definition 5. Preferences represented by $u : A \times \Omega \to \mathbb{R}$ are *weighted Euclidean* if $u(a, \omega) = -l((a - \omega)' W (a - \omega))$, for some $d \times d$ symmetric positive definite matrix W and strictly increasing loss function $l : \mathbb{R}_+ \to \mathbb{R}_+$.

Weighted Euclidean preferences imply that in any state ω, an agent's indifference curves over actions are (rotated) ellipses centered at ω, with actions inside the ellipse more preferred to those outside.[22] A special case is the standard Euclidean preferences, $u(a, \omega) = -\sum_{i=1}^{d} (a_i - \omega_i)^2$, which has spherical indifference curves.

Turning to information, we focus for tractability on the familiar class of *location-shift* information structures: $S = \mathbb{R}^d$ and there is a density $g : \mathbb{R}^d \to \mathbb{R}_{++}$, called the *standard density*, such that $f(s|\omega) = g(s - \omega)$. We restrict attention to standard densities that are uniformly continuous.

Definition 6. A location-shift information structure, or its standard density g, is *subexponential* if there are $p > 1$ and $M > 0$ such that $g(s) < \exp(-\|s\|^p)$ for all $\|s\| > M$.

A subexponential density has a thin tail in the sense that it eventually decays strictly faster than the exponential density. Our leading example of a subexponential location-shift information structure is multivariate normal information: there is some covariance matrix Σ such that the distribution of signals in state ω is $\mathcal{N}(\omega, \Sigma)$. Here the standard density is that of $\mathcal{N}(0, \Sigma)$, and Definition 6 is verified by taking any exponent $p \in (1, 2)$ and any large $M > 0$.

[22] To ensure our maintained assumption of bounded utility, either the function l in Definition 5 must be bounded or the state and action spaces must be bounded.

We can now state:

Proposition 2. *If preferences are weighted Euclidean and the information structure is subexponential location-shift, then there is adequate learning.*

The result stems from Theorem 2, as the combination of subexponential location-shift information and Euclidean preferences yield excludability. To understand why excludability holds, notice first that the preferred sets for Euclidean preferences are *half spaces*, i.e. $\forall a_1 \neq a_2$, $\exists h \in \mathbb{R}^d$ and $c \in \mathbb{R}$ such that $\Omega_{a_1,a_2} = \{\omega : \ h \cdot \omega > c\}$. Excludability then follows from the lemma below.

Lemma 2. *For a subexponential location-shift information structure, $\{\omega : h \cdot \omega \geq c\}$ and $\{\omega : h \cdot \omega < c\}$ are distinguishable from each other for any $h \in \mathbb{R}^d$ and $c \in \mathbb{R}$.*

The exponent p being strictly larger than 1 in the definition of subexponential is essential for the lemma. To see that, consider $d = 1$ and the one-dimensional Laplace or double-exponential standard density, $g(s) = (1/2)\exp(-|s|)$. This density is not subexponential, and indeed the conclusion of Lemma 2 fails: no two states are distinguishable from each other because, for example, if $\omega' < \omega < s$, then $f(s|\omega')/f(s|\omega) = g(s-\omega')/g(s-\omega) = \exp(\omega' - \omega)$ is independent of s.

We can provide an intuition for Lemma 2 by considering a bivariate normal standard density, $g(s) = \exp(-s'\Sigma s/2)/\sqrt{2\pi}$ with Σ a 2×2 covariance matrix. Take an arbitrary hyperplane h, as illustrated in Figure 3. We seek to distinguish the half space to the right of h from its complementary half space to the left. It is sufficient to distinguish an arbitrary single state ω_1 to the right of h from all the states to the left. Figure 3 shows how to construct a sequence of signals verifying that distinguishability. For a sequence of $c_n \to 0$, select s_n so that the iso-density ellipse of level c_n given state ω_1 is tangent with the direction of h at s_n. For all n, the "ellipsoid distance" between s_n and ω_1, $\sqrt{(s_n - \omega_1)'\Sigma(s_n - \omega_1)}$, is then boundedly smaller than the ellipsoid distance between s_n and any state on h, such as ω_2 and ω_3, and hence also (because of the location-shift structure) between s_n and any state to the left of h. Due to the normal distribution being subexponential, as $c_n \to 0$ the likelihood ratio $\frac{g(s_n-\omega)}{g(s_n-\omega_1)} \to 0$ uniformly across ω to the left of h.

Remark 4. In the one-dimensional environment of Subsection 4.1, modulo details about indifferences, SCD is equivalent to preferred sets being half spaces, and DUB is equivalent to the distinguishability of half spaces from their complements in the sense of Lemma 2. This perspective unifies both applications.

Remark 5. A location-shift information structure does not have to be subexponential to guarantee learning for all weighted Euclidean preferences. But it can be shown that if the

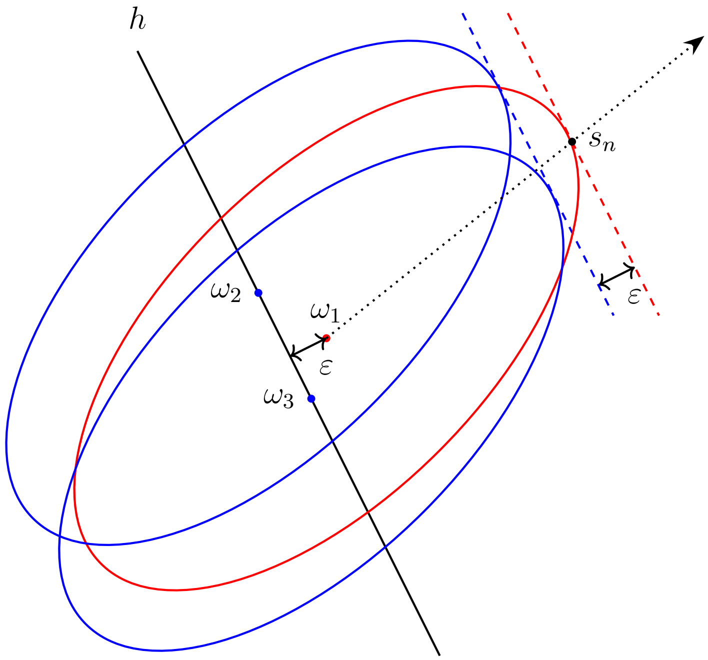

Figure 3: The logic underlying Lemma 2 for a bivariate normal standard density. We seek to distinguish ω_1 from the solid black line. The ellipses are iso-density signals of a given level at the states ω_1, ω_2, and ω_3. As s_n grows along the dotted line, corresponding to lower iso-density levels, $\min\{f(s_n|\omega_2)/f(s_n|\omega_1), f(s_n|\omega_3)/f(s_n|\omega_1)\} \to 0$.

standard density $g : \mathbb{R}^d \to \mathbb{R}$ is superexponential in the sense that there are $p \in (0,1)$ and $M > 0$ such that $g(s) \geq \exp(-\|s\|^p)$ for all $\|s\| > M$, then learning fails for all nontrivial weighted Euclidean preferences.

5. Theorem 1 and a General Welfare Bound

We now return to the general characterization of adequate learning, Theorem 1, to explain how it is derived. The theorem is best understood as a corollary of a welfare bound regardless of whether there is learning. Stating that result requires some notation. Abusing notation, let

$$u(\mu) := \max_{a \in A} \sum_{\omega} u(a,\omega)\mu(\omega)$$

be the maximal utility an agent can get under belief μ. Recalling that μ_s denotes the posterior given a belief μ and signal s, let

$$I(\mu) := \left(\sum_{\omega \in \Omega} \int_S u(\mu_s)\, \mathrm{d}F(s|\omega)\mu(\omega) \right) - u(\mu)$$

be the utility improvement from observing a private signal at belief μ. Observe that $I(\mu) = 0$ for any stationary belief μ. We write $\Phi^{BP} \subset \Delta\Delta\Omega$ to denote the set of Bayes-plausible distributions of beliefs (i.e., $\mathbb{E}_{\varphi}[\mu] = \mu_0 \iff \varphi \in \Phi^{BP}$). Again abusing notation, we write $u(\varphi) := \mathbb{E}_{\varphi}[u(\mu)]$ for the utility of an agent under the distribution of beliefs φ, and analogously write $I(\varphi) := \mathbb{E}_{\varphi}[I(\mu)]$. It follows that

$$\Phi^S := \left\{\varphi \in \Phi^{BP} : I(\varphi) = 0\right\}$$

is the set of Bayes-plausible belief distributions that are supported on the set of stationary beliefs. (We have suppressed the dependence of Φ^{BP} and Φ^S on the prior μ_0.)

Building on a notion mentioned by Lobel and Sadler (2015), we can now define the *cascade utility* level as

$$u_*(\mu_0) := \inf_{\varphi \in \Phi^S} u(\varphi).$$

In words, $u_*(\mu_0)$ is the lowest utility level that an agent can get if her distribution of beliefs is supported on stationary beliefs. Our welfare bound is that eventually all agents are assured a utility level of at least $u_*(\mu_0)$. More precisely:

Theorem 3. *In any equilibrium σ, $\liminf_n \mathbb{E}_{\sigma,\mu_0}[u_n] \geq u_*(\mu_0)$.*

We highlight that the only substantive requirement for this result is our maintained assumption that the observational network structure satisfies expanding observations. It is straightforward to see how Theorem 3 implies the "if" direction of Theorem 1. When all stationary beliefs have adequate knowledge, it holds for any distribution $\varphi \in \Phi^S$ that almost surely a correct action is taken. Thus, $u_*(\mu_0) = u^*(\mu_0)$, and we have adequate learning.

Theorem 3 owes to certain compactness and continuity. First, Lemma 4 in the appendix establishes that Φ^{BP} is compact when both $\Delta\Omega$ and $\Delta\Delta\Omega$ are endowed with the Prohorov metric generated from the discrete metric on Ω. The idea with a countable number of states is that although the prior μ_0 can be supported on an infinite set, it must concentrate an arbitrarily large mass on only finitely many states. Consequently, Bayes-plausibility implies that for any $\delta > 0$, there exists some finite subset $\Omega' \subset \Omega$ such that any $\varphi \in \Phi^{BP}$ must put at least $1 - \delta$ probability on beliefs that put at least $1 - \delta$ probability on Ω', which yields compactness of Φ^{BP}. Second, owing to expected utility being continuous in beliefs, we show that the utility function $u(\varphi)$ and the improvement function $I(\varphi)$ are also continuous (Lemma 5 in the appendix), and thus uniformly continuous on Φ^{BP}. Consequently, $I(\varphi)$ achieves a minimum on any closed, hence compact, subset of Φ^{BP}.

Now consider any ε-neighborhood of the set of Bayes-plausible distributions supported on stationary beliefs, call it $(\Phi^S)^\varepsilon$. If an agent's belief distribution is in $(\Phi^S)^\varepsilon$, then her ex-ante expected utility is at least close to u_* since $I(\varphi) = 0$ on Φ^S and $I(\varphi)$ is uniformly continuous. On the other hand, if the belief distribution is not in $(\Phi^S)^\varepsilon$, then there is some strictly positive minimum utility improvement that the agent obtains (as the complement of $(\Phi^S)^\varepsilon$ is closed).

We can then apply an *improvement principle*, as suggested by Banerjee and Fudenberg (2004) and developed by Acemoglu, Dahleh, Lobel, and Ozdaglar (2011) and others. The idea is as follows, where we consider deterministic networks for simplicity. Expanding observations guarantees that we can partition society into "generations" such that an agent in one generation observes a predecessor who is in either the previous generation or the current generation. We inductively argue that the lowest ex-ante utility in each generation is either close to u_* or increases by a fixed amount compared to the previous generation. Consider an agent's interim belief distribution, φ. Her interim utility $u(\varphi)$ must be at least the lowest ex-ante expected utility of the previous generation, because the current agent can just mimic the agent with the largest index she observes.[23] Then, as explained in the previous paragraph, either $u(\varphi)$ is at least close to u_* (when φ is in $(\Phi^S)^\varepsilon$), or the agent can improve upon $u(\varphi)$ by at least some fixed amount. Thus, the lowest ex-ante expected utility in each generation increases by a fixed amount until it becomes at least close to u_*. Since ε was arbitrary, it follows that eventually all agents' utility must be higher than a level arbitrarily close to u_*, which is the conclusion of Theorem 3.

Although previous authors have deduced versions of Theorem 1 and Theorem 3 in special environments, what allows us to establish these two general results is our novel proof methodology. We highlight two distinctions with Lobel and Sadler (2015, Theorem 1), which is the most related existing result. They consider a binary-state binary-action model. In that setting, they establish a welfare bound of "diffusion utility", which is the utility obtained by a hypothetical agent who observes an information structure that contains only the strongest signals (i.e., an "expert agent", in their terminology). Our cascade utility is more fundamentally tied to when learning stops, as it is defined using stationary beliefs. It is not hard to see that in general, no matter the number of states or actions, cascade utility is always at least as high as (the natural extension of) diffusion utility.[24] As

[23] With stochastic networks, the fact that an agent can obtain any observed predecessor's ex-ante expected utility through mimicking relies on our assumption that players' observation neighborhoods are drawn independently. Otherwise, whether a player has observed some predecessor may correlate with that predecessor realizing a lower utility.

[24] Lobel and Sadler's (2015) definition of diffusion utility is tailored to their binary-binary model. In general, we can define it as the highest utility an agent can obtain from any Bayes-plausible belief distribution

Lobel and Sadler (2015) note, cascade utility and diffusion utility coincide in their binary-state binary-action model; but we note that in general, the former can be strictly higher.[25] Methodologically, Lobel and Sadler's (2015) argument for a minimum improvement, like that of Acemoglu, Dahleh, Lobel, and Ozdaglar (2011), owes to certain monotonicity that does not extend beyond their binary-binary setting. Our approach, by contrast, uses continuity of the improvement function and compactness of the set of Bayes-plausible belief distributions.

Remark 6. Our approach to proving Theorem 3 can be adapted to address belief convergence. Since expanding observations is compatible with the observational network having multiple components, one cannot expect the social belief to converge even in probability.[26] But do social beliefs eventually lie in the stationary set? Yes, under reasonable conditions. Consider deterministic networks and assume that society can be covered by finitely many subsequences such that in each subsequence agent n_k observes n_{k-1}. Then, denoting agent n's (random) social belief by μ_n, it holds that for all $\varepsilon > 0$, $\lim_{n\to\infty} \Pr(\mu_n \in S^\varepsilon) = 1$, where S^ε denotes the ε-neighborhood of the set of stationary beliefs. See Proposition SA.1 in Appendix SA.1. We note that this result applies, in particular, to the immediate-predecessor network and the complete network. The latter is special because the social belief is then a martingale, which is assured to converge almost surely by the martingale convergence theorem. For this case, Arieli and Mueller-Frank (2021, Lemma 1) have established that the limit is stationary.

Remark 7. Theorem 3 can be used to quantify how a failure of excludability impacts welfare. Proposition SA.2 in Appendix SA.2 provides a formal result in this vein. In particular, that result implies a sense in which an environment with close to excludability ensures that, eventually, agents' ex-ante expected utilities are close to the full-information utility.

that is supported on the set of feasible posteriors (i.e., those available in the given information structure and the prior); call the corresponding signal structure the expert signal structure. To see that this utility is lower than cascade utility, notice that this utility must be lower than from first drawing a posterior from an arbitrary Bayes-plausible stationary belief distribution and then drawing a signal from the expert signal structure (as this "combined signal" is Blackwell more informative than just the expert signal); but in the latter, the expert signal has no value by definition of stationary beliefs, and so the combined signal provides a utility equal to that from the (arbitrary) stationary belief distribution.

[25] This is true even with just two states and three actions. Alternatively, consider $\Omega = \{0, 1\}$, $A = [0, 1]$, $u(a, \omega) = -(a - \omega)^2$, and the complete network. It is well known that any nontrivial information structure leads to learning here, with the stationary beliefs being just 0 and 1. So the cascade utility is the full-information utility of 0, whereas diffusion utility will be strictly lower absent unbounded beliefs.

[26] Consider an observational network consisting of two disjoint complete subnetworks: every odd agent observes only all odd predecessors, and symmetrically for even agents. Given any primitives in which learning would fail on a complete network—such as the canonical binary state/binary action herding example—there is positive probability of the limit belief among odd agents being different from that among even agents.

6. Conclusion

This paper has studied a general model of sequential social learning on observational networks. The economic theme underlying our results is that, with multiple states, (asymptotic) learning turns jointly on preferences and information. We close by commenting on certain aspects of our approach.

First, our model assumes "non-anonymous sampling", i.e., whenever an agent sees the action of some predecessor, she knows the identity of that predecessor. However, our methodology extends to anonymous sampling, i.e., when each agent observes only the frequencies of actions in their realized neighborhood, as in Smith and Sørensen (2020). A sufficient condition for our results in this case is that expanding observations, i.e., condition (1), holds for the "induced network structure" $(\tilde{Q}_n)_{n\in\mathbb{N}}$ where each $\tilde{Q}_n$ is defined by first drawing a neighborhood B_n according to Q_n and then uniform-randomly drawing a single agent from B_n. See fn. 30 in the appendix for why this is sufficient. Interestingly, this condition coincides with Smith and Sørensen's (2020) "non-over-sampling" requirement. Note that expanding observations for the induced network $(\tilde{Q}_n)$ is more demanding than expanding observations for (Q_n); this is not surprising since agents have less information when they cannot observe identities. Nevertheless, the requirement is satisfied, for example, when each agent observes the action of a uniform-randomly drawn predecessor or the actions of all predecessors—in either case, not observing their identities. But the requirement is violated when each agent n observes either agent 1 or agent $n-1$, but doesn't observe the identity (whereas expanding observations holds here when the identity is observed).

Second, the notion of learning we have adopted considers all possible priors. While this strengthens our sufficiency results, it correspondingly weakens our necessity results. With only two states, learning at any single (nondegenerate) prior is equivalent to learning at all priors. Our earlier working paper (Kartik, Lee, Liu, and Rappoport, 2022, Supplementary Appendix SA.1) provides some analysis concerning the extent to which this is true with multiple states.

Third, our analysis has not touched on the speed of learning/welfare convergence. For binary states and the complete network, Rosenberg and Vieille (2019) deduce the condition on the likelihood of extreme posteriors that determines whether learning is, in certain senses, efficient; they point out that their condition is violated by normal information. See Hann-Caruthers, Martynov, and Tamuz (2018) as well.

Lastly, our work only addresses Bayesian learning with correctly specified agents. There is a large literature on non-Bayesian social learning, surveyed by Golub and Sadler (2016). There has also been recent interest in (mis)learning among misspecified Bayesian agents; see, for example, Frick, Iijima, and Ishii (2020) and Bohren and Hauser (2021).

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Appendices

A. Backbone Results

We prove our backbone results—Theorem 1, Theorem 2, and Theorem 3—in the following setting, which is more general than that described in the main text.

- The action space and signal space $(A, \mathcal{A}), (S, \mathcal{S})$ are standard Borel spaces;
- The state space Ω is equipped with a metric d and its Borel sigma-algebra, $\mathcal{B}(\Omega)$, such that (Ω, d) is a sigma-compact Polish space;[27]
- The utility function $u(a, \omega)$ has absolute value uniformly bounded by $\overline{u}$ and it is pointwise equicontinuous when regarded as a collection of functions of ω indexed by a; moreover, for every belief (Borel probability distribution over Ω), there exists an optimal action;
- The information/signal structure $F(\cdot|\omega)$ is a Markov kernel from $(\Omega, \mathcal{B}(\Omega))$ to $(S, \mathcal{S})$ that is continuous in ω in the total variation (TV) sense;
- The network structure is given by $Q \equiv (Q_n)_{n\in\mathbb{N}}$, where each Q_n is a probability measure over all neighborhoods, i.e., all subsets of $\{1, 2, \ldots, n-1\}$, independent across n, independent of the state ω, and independent of any private signals.

When Ω is countable as in the main text, we endow it with the discrete metric so that the sigma-compactness and continuity requirements are trivially satisfied.

Discontinuous utilities. While we make a continuity assumption on preferences, our main results hold for utilities satisfying the following condition that permits discontinuities (cf. Remark 1):

Condition 1. There is a countable partition of Ω into Borel sets B_i and pointwise equicontinuous functions $v_i : \Omega \to \mathbb{R}$ uniformly bounded by $\overline{u}$ such that $v_i|_{B_i} = u$.

To obtain our results for such utilities, we can define a new state space $\tilde{\Omega}$ as a disjoint union: $\tilde{\Omega} := \bigsqcup \Omega_i$, where each Ω_i is a copy of Ω. Choose any metric on $\tilde{\Omega}$ that induces the disjoint union topology. Define a utility $\tilde{u}$ on $\tilde{\Omega}$ by $\tilde{u}|_{\Omega_i} := v_i$ for each i. It follows that $\tilde{\Omega}$ is sigma-compact Polish and $\tilde{u}$ is pointwise equicontinuous and uniformly bounded. The

[27] That is, (Ω, d) is a complete and separable metric space that can be represented as a countable union of compact sets.

information structure is defined such that on each Ω_i it is the same as before. Using our results for the new setting, one can deduce Theorems 1–3 for the original setting.[28]

A.1. Overarching Probability Space and Beliefs

We now formalize the overarching probability space over all realizations of the state, signals, observation neighborhoods, and actions. We also define formal objects corresponding to agents' interim and posterior beliefs and distributions of beliefs.

Overarching probability space. Our probability space is constructed from three components: the Markov kernel F and probability space $(\Omega, \mathcal{B}(\Omega), \mu_0)$; the network structure $Q \equiv (Q_n)_{n \in \mathbb{N}}$; each agent n's strategy $\sigma_n(\cdot|a_{B_n}, B_n)$ as a Markov kernel from $(A^{|B_n|}, \mathcal{A}^{|B_n|})$ to $(A, \mathcal{A})$ for each realization of neighborhood B_n.

Taken together, for the first n agents, we can define a probability space that describes the joint distribution of their neighborhoods, signals, actions, and the states. Since all these elements lie in standard Borel spaces, the Kolmogorov Extension Theorem guarantees existence of an overarching probability space $(H_\infty, \mathcal{H}_\infty, \mathbb{P})$ that is consistent with each finite probability space (i.e., up to each agent n). We suppress the dependence of $\mathbb{P}$ on σ and μ_0.

Beliefs. Given this overarching probability space, agent n's interim belief (i.e., her belief after observing her neighbors and their actions, but before observing her private signal) is $\mathbb{P}(\cdot|a_{B_n}, B_n)$ and her posterior belief is $\mathbb{P}(\cdot|a_{B_n}, B_n, s_n)$. These beliefs are well defined because, as a countable product of standard Borel spaces, the overarching probability space is a standard Borel space, and hence there exist regular conditional probabilities (Durrett, 2019, Theorem 4.1.17).

Distribution of beliefs. We denote by $\Delta\Omega$ the space of beliefs (Borel probability measures on Ω) equipped with the Prohorov metric, and by $\Delta\Delta\Omega$ the space of belief distributions (Borel probability measures on $\Delta\Omega$) also equipped with the Prohorov metric.

The interim belief of agent n, μ_n, as a regular conditional probability, can be regarded as a measurable function from $(H_\infty, \mathcal{H}_\infty, \mathbb{P})$ to $(\Delta\Omega, \mathcal{B}(\Delta\Omega))$; see Crauel (2002, Remark 3.20). As Ω is a Polish space, so is $\Delta\Omega$. We define agent n's distribution of (interim) beliefs, φ_n, as the push-forward measure of μ_n. Hence, $\varphi_n \in \Delta\Delta\Omega$ since it is by definition a Borel probability measure on $\Delta\Omega$.

[28] More specifically, the results in the original setting are equivalent to the corresponding results in the new setting restricted to priors/beliefs that put zero probability on the added states $\Omega_i \backslash B_i$. We can use our methodology to derive Theorems 1–3 in the new setting for such restricted beliefs.

A.2. Space of Bayes-Plausible Belief Distributions is Compact

Given a prior $\mu_0 \in \Delta\Omega$ and a strategy profile σ, any agent's belief distribution $\varphi \in \Delta\Delta\Omega$ must be Bayes plausible:

$$\int_A \mu(A)\,\mathrm{d}\varphi(\mu) = \mu_0(A),\ \forall A \in \mathcal{B}(\Omega). \tag{2}$$

Let $\Phi^{BP} \subset \Delta\Delta\Omega$ be the set of Bayes-plausible belief distributions; note that we suppress the dependence of Φ^{BP} on μ_0.

Our goal is to establish (Lemma 4 below) that even though the set of belief distributions $\Delta\Delta\Omega$ need not be compact, the subset of Bayes-plausible distributions Φ^{BP} is. A key step is the following lemma, which shows that any belief $\varphi \in \Phi^{BP}$ has to put a large probability on a compact subset of $\Delta\Omega$.

Lemma 3. *Let $\delta > 0$ and $\{\Omega_i\}_{i\in\mathbb{N}}$ be a sequence of compact sets with $\mu_0(\Omega_i) \geq 1 - (\frac{\delta}{2^i})^2$, $\forall i$. Defining $V_\delta := \{\mu \in \Delta\Omega : \mu(\Omega_i) \geq 1 - \frac{\delta}{2^i},\ \forall i\}$, it holds that:*

1. *V_δ is compact;*
2. *$\varphi(\mu \notin V_\delta) < \delta, \forall \varphi \in \Phi^{BP}$.*

Intuitively, in the lemma's statement, the set V_δ contains all beliefs that put high probability on a set of states that the prior μ_0 ascribes high probability to. The lemma concludes that the set V_δ is compact and that any Bayes-plausible belief distribution must put high probability on V_δ.

Proof. (Part 1) First, V_δ is closed. To see this, take any $\mu_k \to \mu$ and $\mu_k \in V_\delta$. Since each Ω_i is compact (and thus closed), weak convergence implies

$$\limsup_k \mu_k(\Omega_i) \leq \mu(\Omega_i),\ \forall i,$$

which implies $\mu(\Omega_i) \geq 1 - \frac{\delta}{2^i}$. Thus, $\mu \in V_\delta$, and hence V_δ is closed.

Next, the beliefs in V_δ are tight by definition. Hence, by Prohorov's theorem, the closure of V_δ, which is V_δ itself, is compact.

(Part 2) Note that $\varphi(\mu \notin V_\delta) = \varphi(\cup_i\{\mu(\Omega_i^c) > \frac{\delta}{2^i}\}) \leq \sum_i \varphi(\mu(\Omega_i^c) > \frac{\delta}{2^i})$. For each $i \in \mathbb{N}$, we view $\mu(\Omega_i^c)$ as a non-negative random variable with a distribution φ. Since φ is Bayes plausible, $\mathbb{E}_\varphi[\mu(\Omega_i^c)] = \mu_0(\Omega_i^c) \leq (\frac{\delta}{2^i})^2$, which implies (using Markov's inequality) that $\varphi(\mu(\Omega_i^c) > \frac{\delta}{2^i}) < \frac{\delta}{2^i}$. This implies that $\varphi(\mu \notin V_\delta) < \sum_i \frac{\delta}{2^i} = \delta$. *Q.E.D.*

Given Lemma 3, we can use Prohorov's theorem again to Φ^{BP} and obtain:

Lemma 4. Φ^{BP} *is compact.*

Proof. First, to prove that Φ^{BP} is closed, it is sufficient to take any $\varphi_k \to \varphi$ and $\varphi_k \in \Phi^{BP}$, and show that $\varphi \in \Phi^{BP}$, i.e., $\mathbb{E}_\varphi[\mu(W)] = \mu_0(W), \forall W \in \mathcal{B}(\Omega)$.

Take any open set $W \in \mathcal{B}(\Omega)$. For any $\mu_k \to \mu$, it holds that $\mu(W) \leq \liminf \mu_k(W)$. In other words, $\mu(W)$ (as a function of μ) is lower semi-continuous. By properties of weak convergence, it follows that $\mathbb{E}_\varphi[\mu(W)] \leq \liminf \mathbb{E}_{\varphi_k}[\mu(W)] = \mu_0(W)$. That is, the mean measure of φ ascribes a smaller probability than μ_0 to any open set.

Now observe that $W^c \subseteq \cup_{x\in W^c} B_{1/n}(x)$ for any n. Hence,

$$\mathbb{E}_\varphi[\mu(W^c)] \leq \lim_n \mathbb{E}_\varphi[\mu(\cup_{x\in W^c} B_{1/n}(x))] \leq \lim_n \mu_0(\cup_{x\in W^c} B_{1/n}(x)) = \mu_0(W^c),$$

where the second inequality is from the previous result applied to open sets $\cup_{x\in W^c} B_{1/n}(x)$, and the last equality follows from $W^c = \cap_n \cup_{x\in W^c} B_{1/n}(x)$ (and this equality holds because W^c is closed). Therefore, $\mathbb{E}_\varphi[\mu(W)] = \mu_0(W)$.

Since $\mathbb{E}_\varphi[\mu]$ and μ_0 agree on all open sets, and open sets generate $\mathcal{B}(\Omega)$, $\mathbb{E}_\varphi[\mu]$ and μ_0 agree on all sets in $\mathcal{B}(\Omega)$.

Finally, Ω being sigma-compact implies that for any δ, there is an increasing sequence of compact sets $\{\Omega_i\}_{i\in\mathbb{N}}$ such that $\Omega = \cup_i \Omega_i$, and so this sequence $\{\Omega_i\}$ satisfies the hypotheses in Lemma 3. The lemma guarantees that there is a compact set V_δ such that $\varphi(V_\delta) < \delta$ for all $\varphi \in \Phi^{BP}$, and so Φ^{BP} is tight. Prohorov's theorem now implies that $\mathrm{cl}(\Phi^{BP}) = \Phi^{BP}$ is compact. *Q.E.D.*

A.3. Continuity of Various Functions

We next define some functions of interest, some of which were already defined in the main text but are now defined for the more general setting considered in the appendix.

Let $u(\mu)$ be the expected utility that an agent can get at belief μ:

$$u(\mu) := \sup_a \int_\Omega u(a, \omega) \, \mathrm{d}\mu(\omega).$$

Let $u^F(\mu)$ be the expected utility that an agent can get at belief μ, if she can choose an action after observing her private signal:

$$u^F(\mu) := \sup_{\beta: S\to A} \int_\Omega \int_S u(\beta(s), \omega) \, \mathrm{d}F(s|\omega) \, \mathrm{d}\mu(\omega).$$

Finally, let $u^*(\mu)$ be the full information utility at μ:

$$u^*(\mu) := \int_\Omega \sup_a u(a,\omega)\,\mathrm{d}\mu(\omega).$$

Our continuity assumptions on the utility function and the information structure allow us to prove:

Lemma 5. *u, u^F, u^* are continuous in μ.*

To prove Lemma 5, we require Theorem 2.2.8 in Bogachev (2018), which we restate without proof for our context as the following claim:

Claim 1. *Let $\mu_k \to \mu$. If Γ is a uniformly bounded and pointwise equicontinuous family of functions on Ω, then*

$$\lim_k \sup_{f\in\Gamma} \left| \int_\Omega f\,\mathrm{d}\mu_k - \int_\Omega f\,\mathrm{d}\mu \right| = 0.$$

Proof of Lemma 5. By assumption, $\Gamma := \{u(a,\omega)\}_{a\in A}$, viewed as a family of functions of ω indexed by a, is uniformly bounded and pointwise equicontinuous.

Consider the function u^*. Since the supremum of the pointwise equicontinuous functions $u^*(\omega) := \sup_a u(a,\omega)$ is continuous in ω, the definition of weak convergence implies that $u^*(\mu)$ is continuous in μ.

Now consider the function u. Its continuity follows from

$$|u(\mu_k) - u(\mu)| = \left| \sup_{f\in\Gamma} \int_\Omega f\,\mathrm{d}\mu_k - \sup_{f\in\Gamma} \int_\Omega f\,\mathrm{d}\mu \right| \le \sup_{f\in\Gamma} \left| \int_\Omega f\,\mathrm{d}\mu_k - \int_\Omega f\,\mathrm{d}\mu \right|,$$

which converges to 0 for $\mu_k \to \mu$ by Claim 1.

Lastly, suppose we establish that $\Gamma^F := \left\{ \int_S u(\beta(s),\omega)\,\mathrm{d}F(s|\omega) \right\}_{\beta: S\to A}$, as a family of functions of ω indexed by β, is pointwise equicontinuous.[29] Then, as Γ^F is uniformly bounded, Claim 1 implies that $u^F(\mu)$ is continuous, proving the lemma.

To establish the pointwise equicontinuity of Γ^F, observe that $\forall \omega, \omega'$ and $\forall \beta$,

$$\begin{aligned} &\left| \int_S u(\beta(s),\omega)\,\mathrm{d}F(s|\omega) - \int_S u(\beta(s),\omega')\,\mathrm{d}F(s|\omega') \right| \\ \le &\left| \int_S (u(\beta(s),\omega) - u(\beta(s),\omega'))\,\mathrm{d}F(s|\omega) \right| + \left| \int_S u(\beta(s),\omega')\,\mathrm{d}F(s|\omega) - \int_S u(\beta(s),\omega')\,\mathrm{d}F(s|\omega') \right|. \end{aligned} \tag{3}$$

[29] Here we assume β are (measurable) pure strategies for notation clarity. The same argument works for mixed strategies, in which case β would be Markov kernels.

Fix any ω and any $\varepsilon > 0$. Since $\{u(a, \omega)\}_{a \in A}$ is pointwise equicontinuous, there exists δ_1 such that $d(\omega', \omega) < \delta_1$ implies the first term on the right-hand side of inequality (3) to be smaller than $\varepsilon/2$ (regardless of $\beta(s)$). The second term is smaller than $2\overline{u} d_{TV}(F(\cdot|\omega), F(\cdot|\omega'))$ (where TV represents total variation), and by the continuity assumption of the information structure, there exists $\delta_2 > 0$ such that $d(\omega', \omega) < \delta_2$ implies $d_{TV}(F(\cdot|\omega), F(\cdot|\omega')) < \varepsilon/4\overline{u}$. Therefore, if $d(\omega', \omega) < \min\{\delta_1, \delta_2\}$, then the right-hand side of inequality (3) is less than ε (regardless of $\beta(s)$). It follows that Γ^F is pointwise equicontinuous. *Q.E.D.*

Now define the utility improvement $I(\mu)$ and the utility gap $G(\mu)$ at μ as:

$$I(\mu) := u^F(\mu) - u(\mu), \quad G(\mu) := u^*(\mu) - u(\mu).$$

By Lemma 5, $I(\mu)$ and $G(\mu)$ are continuous. Lastly, with an abuse of notation, define $u(\varphi) := \mathbb{E}_\varphi[u(\mu)]$, $I(\varphi) := \mathbb{E}_\varphi[I(\mu)]$, and $G(\varphi) := \mathbb{E}_\varphi[G(\mu)]$ as the corresponding functions over distributions of beliefs. Since $u(\mu)$, $I(\mu)$, and $G(\mu)$ are continuous, so are $u(\varphi)$, $I(\varphi)$, and $G(\varphi)$.

A.4. Proofs for Backbone Results

We say that a belief μ is *stationary* if $I(\mu) = 0$, and a belief μ has *adequate knowledge* if $G(\mu) = 0$. These definitions agree with those in the main text. To confirm that, consider stationary beliefs. If there is an action that is a.s. optimal regardless of the signal, then clearly $I(\mu) = 0$. Conversely, if there is no action that is a.s. optimal regardless of the signal, then for any action there is a positive-probability set of signals for which that action is strictly suboptimal; hence $u^F(\mu) > u(\mu)$, and $I(\mu) > 0$. The argument for adequate knowledge beliefs is similar.

Logically, Theorem 3 $\implies$ Theorem 1 $\implies$ Theorem 2. So we prove the results in that order.

Proof of Theorem 3. We prove the result in two steps. In Step 1 below, we prove that if agent n's belief distribution φ_n, which is her belief distribution incorporating the observation of her neighborhood's actions but not her private signal, is not close to being supported on only stationary beliefs, then her utility $\mathbb{E}_{\sigma,\mu_0}[u_n]$, which is the ex-ante expected utility under equilibrium σ after observing the private signal, improves from $u(\varphi_n)$ by a fixed positive minimum. In Step 2 below, we use the expanding observations assumption to establish that this minimum improvement propagates through the network until eventually agents obtain at least arbitrarily close to their cascade utility level.

Step 1: Recall the set of Bayes-plausible belief distributions that are supported by stationary beliefs, $\Phi^S := \{\varphi \in \Phi^{BP} : I(\varphi) = 0\}$, and the *cascade utility*, $u_* := \inf_{\varphi \in \Phi^S} u(\varphi)$.

Take any $\varepsilon > 0$, and let $(\Phi^S)^\varepsilon$ denote the ε-neighborhood of Φ^S. An agent n's belief distribution φ_n must be Bayes plausible, so $\varphi_n \in \Phi^{BP}$. Since $u(\varphi)$ is uniformly continuous (being continuous on the compact set Φ^{BP}), if $\varphi_n \in (\Phi^S)^\varepsilon$, then $u(\varphi_n) \geq u_* - \gamma(\varepsilon)$ for some $\gamma(\cdot)$ such that $\gamma(\varepsilon) \to 0$ when $\varepsilon \to 0$. If, on the other hand, $\varphi_n \in \Phi^+ := \Phi^{BP} \backslash (\Phi^S)^\varepsilon$, then $I(\varphi_n) > 0$ because φ_n puts positive probability on $\{\mu : I(\mu) > 0\}$. Since $(\Phi^S)^\varepsilon$ is open, Φ^+ is a closed subset of a compact set Φ^{BP}; hence Φ^+ is compact, and since $I(\varphi)$ is continuous, it attains a minimum over Φ^+ at some $\underline{\varphi} \in \Phi^+$. Thus, if $\varphi_n \in \Phi^+$ the agent obtains a minimum improvement $I(\varphi_n) \geq I(\underline{\varphi}) > 0$.

Step 2: We will argue that for any $\varepsilon > 0$, $\mathbb{E}_{\sigma,\mu_0}[u_n] \geq u_* - \gamma(\varepsilon)$ once n is large enough. Since ε is arbitrary, taking $\varepsilon \to 0$ implies $\liminf_n \mathbb{E}_{\sigma,\mu_0}[u_n] \geq u_*$, which completes the proof.

For a given $\varepsilon > 0$, let $\delta = \frac{I(\underline{\varphi})}{4\overline{u}} > 0$, let $N_0 = 1$, and define N_k for $k = 1, 2, \ldots$ sequentially such that for all $n \geq N_k$, $Q_n(\max_{b \in B_n} b < N_{k-1}) < \delta$. Expanding observations ensures that such N_k exist.

We claim that, for any agent $n \geq N_k$, $\mathbb{E}_{\sigma,\mu_0}[u_n] \geq \alpha_k := \min\{u_* - \gamma(\varepsilon), \frac{kI(\underline{\varphi})}{2} - \overline{u}\}$. Since $\alpha_0 = -\overline{u}$, clearly $\mathbb{E}_{\sigma,\mu_0}[u_n] \geq \alpha_0$ for any $n \geq N_0$. Suppose the claim holds for all agents $n' \geq N_{k-1}$. Take any agent $n \geq N_k$. Agent n's neighborhood is drawn independently of everything that has happened before, so conditional on agent n observing an agent $n' \geq N_{k-1}$, even without her private signal agent n can achieve a utility of at least α_{k-1} by imitating agent n'. Hence, $u(\varphi_n) \geq (1-\delta) \cdot \alpha_{k-1} + \delta \cdot (-\overline{u})$.[30] If $\varphi_n \in (\Phi^S)^\varepsilon$, then by definition $u(\varphi_n) \geq u_* - \gamma(\varepsilon)$, and thus $\mathbb{E}_{\sigma,\mu_0}[u_n] \geq u(\varphi_n) \geq u_* - \gamma(\varepsilon) \geq \alpha_k$. If $\varphi_n \notin (\Phi^S)^\varepsilon$, then agent n can improve her utility by at least $I(\underline{\varphi})$, and so

$$\begin{aligned}
\mathbb{E}_{\sigma,\mu_0}[u_n] &\geq (1-\delta)\alpha_{k-1} + \delta \cdot (-\overline{u}) + I(\underline{\varphi}) \\
&\geq \alpha_{k-1} + \frac{I(\underline{\varphi})}{2} \quad \text{(because } \alpha_{k-1} \leq \overline{u} \text{ and } \delta = \frac{I(\underline{\varphi})}{4\overline{u}}\text{)} \\
&\geq \alpha_k.
\end{aligned}$$

Since the definition of α_k implies that there is a finite K such that for all $k \geq K$, $\alpha_k = u_* - \gamma(\varepsilon)$, it follows that for all $n \geq N_K$, $\mathbb{E}_{\sigma,\mu_0}[u_n] \geq u_* - \gamma(\varepsilon)$. *Q.E.D.*

[30] If agents do not observe the identities associated with the observed actions of their predecessors, an agent can uniform-randomly select one of the actions they observe to imitate. So long as the "induced network structure" (i.e., a network structure $(\tilde{Q}_n)$ wherein each $\tilde{Q}_n$ is defined by first drawing a neighborhood B_n from Q_n and then uniform-randomly drawing a single agent from B_n) satisfies expanding observations, the current proof goes through without change using the induced network structure.

Proof of Theorem 1. The "only if" direction is straightforward. If there is a stationary belief without adequate knowledge, then when the prior is that belief there is an equilibrium where each agent ignores her signal and action history and obtains a utility that is strictly below the full-information utility level.

For the "if" direction, fix any prior μ_0 and equilibrium σ. Since all stationary beliefs have adequate knowledge, $I(\mu) = 0$ implies $G(\mu) = 0$. Thus, for any $\varphi \in \Phi^S$, $\varphi(\{\mu : I(\mu) = 0\}) = \varphi(\{\mu : G(\mu) = 0\}) = 1$, which implies $G(\varphi) = u^*(\varphi) - u(\varphi) = 0$. Moreover, because μ_0 is the mean measure of φ,

$$u^*(\varphi) = \mathbb{E}_\varphi \left[\int_\Omega \sup_a u(a,\omega) \, \mathrm{d}\mu \right] = \int_\Omega \sup_a u(a,\omega) \, \mathrm{d}\mu_0 = u^*(\mu_0),$$

which implies $u(\varphi) = u^*(\mu_0)$. As a result, $u_*(\mu_0) = \inf_{\varphi \in \Phi^S} u(\varphi) = u^*(\mu_0)$. It follows from Theorem 3 that $\liminf_n \mathbb{E}_{\sigma,\mu_0}[u_n] \geq u^*(\mu_0)$. Since $\mathbb{E}_{\sigma,\mu_0}[u_n] \leq u^*(\mu_0)$ for all n, it further follows that $\mathbb{E}_{\sigma,\mu_0}[u_n] \to u^*(\mu_0)$. As μ_0 and σ are arbitrarily, we have adequate learning. *Q.E.D.*

Next we state and prove a more general version of Theorem 2. For any $n \in \mathbb{N}$, define $\Omega^n_{a_1,a_2} := \{\omega : u(a_1,\omega) - u(a_2,\omega) > \frac{1}{n}\}$.

Theorem 2′. *Excludability implies adequate learning at every choice set. There is inadequate learning for choice set $\{a_1, a_2\}$ if Ω_{a_1,a_2} is not distinguishable from $\Omega^n_{a_2,a_1}$ for some n.*

Note that when Ω is finite, or the utility difference between any pair of actions is bounded away from zero, a failure of excludability is equivalent to the condition for necessity in the theorem holding for some a_1, a_2. Hence Theorem 2 is implied by Theorem 2′.

Proof of Theorem 2′. (First statement) First note that excludability (under the full choice set A) implies excludability under any choice subset $A' \subseteq A$. So we fix an arbitrary $A' \subseteq A$ and show that excludability under that subset implies adequate learning at that choice subset. In what follows, the domain of actions should be understood as A', and we denote a typical element by a'.

Theorem 1 implies that we need only show that any $\mu \in \Delta\Omega$ with inadequate knowledge is not stationary. So take any $\mu \in \Delta\Omega$ with inadequate knowledge and any $a^* \in c(\mu)$. Since there is inadequate knowledge, $\mu(\cup_{a'}\Omega_{a',a^*}) > 0$, i.e., there is a positive measure of states where a^* is not optimal. The continuity of $u(a',\omega) - u(a^*,\omega)$ implies that Ω^n_{a',a^*} are open sets for any a' and n. Since Ω is Polish, it is second-countable and hence has a countable basis. Therefore, each open set Ω^n_{a',a^*}, and hence the open set $\cup_{a'}\Omega_{a',a^*} (= \cup_{a'} \cup_n \Omega^n_{a',a^*})$, is a union of countably many basic open sets. Since $\mu(\cup_{a'}\Omega_{a',a^*}) > 0$, at least one basic open set contained in Ω^n_{a',a^*} for some a' and n has strictly positive measure, i.e., $\mu(\Omega^n_{a',a^*}) > 0$.

Now denote $\mu'(\cdot) := \mu(\cdot|\Omega_{a^*,a'} \cup \Omega^n_{a',a^*})$ as the corresponding conditional probability. Since Ω_{a',a^*} is distinguishable from $\Omega_{a^*,a'}$ by excludability, so is Ω^n_{a',a^*}.[31] Therefore, for any $\varepsilon > 0$ there exists a set of signals S' such that $\mathbb{P}_{\mu'}(S') > 0$ and $\mu'_s(\Omega^n_{a',a^*}) > 1-\varepsilon$ for all $s \in S'$. The utility improvement upon observing any $s \in S'$ by switching from a^* to a' is therefore bounded below by $(\frac{1}{n}(1-\varepsilon) - 2\overline{u}\varepsilon)\mu_s(\Omega_{a^*,a'} \cup \Omega^n_{a',a^*})$, as the expected improvement on $\Omega \backslash (\Omega_{a^*,a'} \cup \Omega^n_{a',a^*})$ is nonnegative. For small $\varepsilon > 0$, $\frac{1}{n}(1-\varepsilon) - 2\overline{u}\varepsilon > 0$, and hence, integrating over S', the ex-ante improvement is bounded below by $(\frac{1}{n}(1-\varepsilon) - 2\overline{u}\varepsilon)\mathbb{P}_{\mu'}(S')\mu(\Omega_{a^*,a'} \cup \Omega^n_{a',a^*}) > 0$. It follows that $I(\mu) > 0$, and thus μ is not stationary.

(Second statement) Suppose there are two actions a_1, a_2 and an n such that Ω_{a_1,a_2} is not distinguishable from $\Omega^n_{a_2,a_1}$. This means there exists $\mu \in \Delta(\Omega_{a_1,a_2} \cup \Omega^n_{a_2,a_1})$ with $\mu(\Omega_{a_1,a_2}) > 0$ such that $\mu_s(\Omega_{a_1,a_2}) \leq 1-\varepsilon$ for some $\varepsilon > 0$ and μ-a.e. s. Consider $\mu' \in \Delta(\Omega_{a_1,a_2} \cup \Omega^n_{a_2,a_1})$ with a small $\mu'(\Omega_{a_1,a_2}) > 0$ such that $\mu'(\cdot|\Omega_{a_1,a_2}) = \mu(\cdot|\Omega_{a_1,a_2})$ and $\mu'(\cdot|\Omega^n_{a_2,a_1}) = \mu(\cdot|\Omega^n_{a_2,a_1})$. Under μ', upon observing signal s, the posterior on Ω_{a_1,a_2} satisfies

$$\frac{\mu'_s(\Omega_{a_1,a_2})}{\mu'_s(\Omega^n_{a_2,a_1})} = \frac{\mu_s(\Omega_{a_1,a_2})/\mu(\Omega_{a_1,a_2})}{\mu_s(\Omega^n_{a_2,a_1})/\mu(\Omega^n_{a_2,a_1})}\frac{\mu'(\Omega_{a_1,a_2})}{\mu'(\Omega^n_{a_2,a_1})} \leq \frac{1-\varepsilon}{\varepsilon}\frac{\mu(\Omega^n_{a_2,a_1})}{\mu(\Omega_{a_1,a_2})}\frac{\mu'(\Omega_{a_1,a_2})}{\mu'(\Omega^n_{a_2,a_1})}$$

for μ-a.e. s. Hence, by choosing μ' so that $\frac{\mu'(\Omega_{a_1,a_2})}{\mu'(\Omega^n_{a_2,a_1})}$ is arbitrarily small, the ratio $\frac{\mu'_s(\Omega_{a_1,a_2})}{\mu'_s(\Omega^n_{a_2,a_1})}$ can be made arbitrarily small uniformly over s.

Under μ', after observing s, the expected improvement by switching from a_2 to a_1 is bounded above by $2\overline{u}\mu'_s(\Omega_{a_1,a_2}) - \frac{1}{n}\mu'_s(\Omega^n_{a_2,a_1})$, which is strictly negative when $\frac{\mu'_s(\Omega_{a_1,a_2})}{\mu'_s(\Omega^n_{a_2,a_1})}$ is small. Therefore, for μ'-a.e. s, a_2 is strictly better than a_1, and thus μ' is stationary for choice set $\{a_1, a_2\}$. However, since $\mu'(\Omega_{a_1,a_2}) > 0$, the belief μ' has inadequate knowledge. Theorem 1 implies there is inadequate learning for choice set $\{a_1, a_2\}$. *Q.E.D.*

B. Applications

We now specialize to the main text's setting: Ω is countable, endowed with the discrete metric, and $F(\cdot|\omega)$ are absolutely continuous with respect to each to other, and so there are densities $f(\cdot|\omega) > 0$.

[31] In fact, excludability is equivalent to: $\Omega^n_{a_1,a_2}$ is distinguishable from Ω_{a_2,a_1} for all a_1, a_2 and n.

B.1. SCD Preferences & DUB Information

Proof of Proposition 1. Sufficiency follows directly from Theorem 2. For necessity, first observe that if the information structure fails DUB, then there exists some state ω^* such that ω^* is not distinguishable from its lower set (or from its upper set, which has a symmetric argument). Fix any pair of distinct actions a_1 and a_2, and define the following SCD preferences: for $\omega < \omega^*$, $u(a_1, \omega) = 1$ and $u(a_2, \omega) = 0$; for $\omega \geq \omega^*$, $u(a_1, \omega) = 0$ and $u(a_2, \omega) = 1$; and any other actions are strictly dominated. It follows that Ω_{a_2,a_1} is not distinguishable from $\{\omega : u(a_1, \omega) - u(a_2, \omega) > \frac{1}{2}\}$. By Theorem 2′, there is inadequate learning when the choice is $\{a_1, a_2\}$, and since all other actions are strictly dominated, also for the full choice set A. *Q.E.D.*

B.2. Euclidean Preferences & Location-Shift Information

The proof of Lemma 2 is more involved than the intuition given in the main text using Figure 3, because in general one cannot explicitly identify the sequence of signals that establishes distinguishability of the relevant two sets.

We will use the following claim in proving Lemma 2. For any $h, x \in \mathbb{R}^d$ and $c \in \mathbb{R}$, let $\|x\|_h := h \cdot x - c$ be the "signed distance" between x and the hyperplane $\{z : h \cdot z = c\}$.

Claim 2. *If a standard density g is subexponential, then for any $\bar{s}$ with $\|\bar{s}\|_h > 0$, and $\varepsilon \in (0, 1)$, there is s with $\|s - \bar{s}\|_h \geq 1$ such that:*

1. $\sup_{\{s' : \|s'-s\|_h \geq 1/\|\bar{s}\|_h\}} \frac{g(s')}{g(s)} < \varepsilon$*; and*
2. $\sup_{\{s' : 0 < \|s'-s\|_h < 1/\|\bar{s}\|_h\}} \frac{g(s')}{g(s)} < 2.$

Proof. Suppose not, to contradiction. Then there exists $\bar{s}$ with $\|\bar{s}\|_h > 0$ and $\varepsilon \in (0, 1)$ with the following property: for every s with $\|s - \bar{s}\|_h \geq 1$, we can find s' with $\|s' - s\|_h > 0$ such that either (i) $\|s' - s\|_h \geq 1/\|\bar{s}\|_h$ and $\frac{g(s')}{g(s)} \geq \varepsilon$, or (ii) $0 < \|s' - s\|_h < 1/\|\bar{s}\|_h$ and $\frac{g(s')}{g(s)} \geq 2$. For an arbitrary choice of s' given s, we define $k_s := \|s' - s\|_h$. That means, for each s with $\|s - \bar{s}\|_h \geq 1$, we have $k_s > 0$ and a signal s' with $\|s' - s\|_h = k_s$ such that either (i) $\frac{g(s')}{g(s)} \geq \varepsilon \geq \varepsilon^{k_s \|\bar{s}\|_h}$ (because $k_s \|\bar{s}\|_h \geq 1$), or (ii) $\frac{g(s')}{g(s)} \geq 2 > \varepsilon^{k_s \|\bar{s}\|_h}$ (because $\varepsilon < 1$).

We construct a sequence of signals $(s_i)_{i=1}^{\infty}$. First, take any s_1 such that $\|s_1 - \bar{s}\|_h = 1$. Then, for all $i > 1$, take any s_i given s_{i-1} as explained in the previous paragraph. Note that for all i, $\|s_i - s_{i-1}\|_h = k_{s_{i-1}}$, so $\|s_i\|_h = (\|\bar{s}\|_h + 1) + \sum_{j=1}^{i-1} k_{s_j}$.

First, suppose that $\sum_{i=1}^{\infty} k_{s_i} = \infty$, so that $\lim_{i \to \infty} \|s_i\|_h = \infty$. It holds that for all s_i,

$\frac{g(s_i)}{g(\bar{s})} \geq \frac{g(s_1)}{g(\bar{s})}\varepsilon^{(k_{s_{i-1}}+\cdots+k_{s_1})\|\bar{s}\|_h} = \frac{g(s_1)}{g(\bar{s})}\varepsilon^{(\|s_i\|_h-\|\bar{s}\|_h-1)\|\bar{s}\|_h}$, which in turn implies that

$$(\|s_i\|_h - \|\bar{s}\|_h - 1)\|\bar{s}\|_h \log(\varepsilon) + \log(g(s_1)) \leq \log(g(s_i)). \tag{4}$$

However, since g is subexponential, and $\|s_i\|_h$ and $\|s_i\|$ are of the same order when $\|s_i\|_h$ is large, there is $p > 1$ such that for all large enough i,

$$\log(g(s_i)) < -(\|s_i\|_h)^p. \tag{5}$$

The left-hand side of inequality (4) is linear in $\|s_i\|_h$ while the right-hand side of inequality (5) has exponent $p > 1$, so for large enough i these inequalities are in contradiction.

Next, suppose instead $\lim_{i\to\infty}\|s_i\|_h < \infty$. Then there is N such that for all $i \geq N$, we have $k_{s_i} < 1/\|\bar{s}\|_h$ and thus $\frac{g(s_{i+1})}{g(s_i)} \geq 2$. It follows that $\lim_{i\to\infty}\frac{g(s_i)}{g(s_N)} \geq \lim_{i\to\infty} 2^{i-N} = \infty$. This contradicts the boundedness of g (being a density, g is bounded because it is uniformly continuous). *Q.E.D.*

Proof of Lemma 2. We only prove that $\{\omega : h\cdot\omega \geq c\}$ is distinguishable from $\{\omega : h\cdot\omega < c\}$. The other direction is similar because any state in $\{\omega : h\cdot\omega < c\}$ has a neighborhood that is strictly separated from $\{\omega : h\cdot\omega \geq c\}$.

We use Claim 2 iteratively to construct a signal sequence $(s_i^*)_{i=1}^\infty$. Choose any s_1^* with $\|s_1^*\|_h > 0$, and for $i > 1$, choose any s_i^* such that $\|s_i^* - s_{i-1}^*\|_h \geq 1$ that satisfies (i) $\sup_{\{s':\|s'-s_i^*\|_h \geq 1/\|s_{i-1}^*\|_h\}}\frac{g(s')}{g(s_i^*)} < \frac{1}{i-1}$ and (ii) $\sup_{\{s':0<\|s'-s_i^*\|_h<1/\|s_{i-1}^*\|_h\}}\frac{g(s')}{g(s_i^*)} < 2$. This construction is well-defined by Claim 2, with $\lim_{i\to\infty}\|s_i^*\|_h = \infty$.

As noted after Definition 1, it is sufficient to prove that any $\bar{\omega} \in \{\omega : h\cdot\omega \geq c\}$ is distinguishable from $\{\omega : h\cdot\omega < c\}$.[32] So take any such $\bar{\omega}$ and μ with $\mu(\bar{\omega}) > 0$. Define $\bar{s}_i := s_i^* + \bar{\omega}$. It follows that for all i,

$$\|\omega - \bar{\omega}\|_h < 0 \implies \frac{f(\bar{s}_i|\omega)}{f(\bar{s}_i|\bar{\omega})} = \frac{g(\bar{s}_i - \omega)}{g(\bar{s}_i - \bar{\omega})} = \frac{g(s_i^* + (\bar{\omega}-\omega))}{g(s_i^*)} < 2,$$

and

$$\|\omega - \bar{\omega}\|_h \leq -\frac{1}{\|s_{i-1}^*\|_h} \implies \frac{f(\bar{s}_i|\omega)}{f(\bar{s}_i|\bar{\omega})} = \frac{g(s_i^* + (\bar{\omega}-\omega))}{g(s_i^*)} < \frac{1}{i-1},$$

[32] We note that this uses the assumption of countable states.

and thus,

$$\frac{\mu(\{\omega : h\cdot\omega < c\}|\overline{s}_i)}{\mu(\overline{\omega}|\overline{s}_i)} \leq \frac{\sum_{\|\omega-\overline{\omega}\|_h<0}\mu(\omega)f(\overline{s}_i|\omega)}{\mu(\overline{\omega})f(\overline{s}_i|\overline{\omega})}$$
$$< \frac{1}{i-1}\frac{\sum_{\|\omega-\overline{\omega}\|_h\leq -1/\|s^*_{i-1}\|_h}\mu(\omega)}{\mu(\overline{\omega})} + 2\frac{\sum_{-1/\|s^*_{i-1}\|_h<\|\omega-\overline{\omega}\|_h<0}\mu(\omega)}{\mu(\overline{\omega})}.$$

The last expression can be taken arbitrarily small because $\|s^*_{i-1}\|_h \to \infty$ as $i\to\infty$. Since g is uniformly continuous, there is a neighborhood $\overline{S}_i$ around $\overline{s}_i$ over which the last inequality above continues to hold. Therefore, $\overline{\omega}$ is distinguishable from $\{\omega : h\cdot\omega < c\}$. *Q.E.D.*

B.3. Other Proofs

Proof of Lemma 1. As noted before the lemma, Ω' is distinguishable from Ω'' if and only if each $\omega' \in \Omega'$ is distinguishable from Ω''. So fix any $\omega' \in \Omega'$.

We first prove that if the lemma's condition holds, then ω' is distinguishable from Ω''. Take any probability measure $\mu \in \Delta(\{\omega'\}\cup\Omega'')$ such that $\mu(\omega') > 0$. By assumption, for any $\varepsilon > 0$ there exists a positive-probability set of signals S' such that $\frac{f(s|\omega'')}{f(s|\omega')} < \varepsilon, \forall\omega'' \in \Omega'', \forall s \in S'$. It follows that for all $s \in S'$,

$$\mu(\omega'|s) = \frac{f(s|\omega')\mu(\omega')}{\sum_{\tilde{\omega}\in\{\omega'\}\cup\Omega''} f(s|\tilde{\omega})\mu(\tilde{\omega})} = \frac{\mu(\omega')}{\mu(\omega') + \sum_{\tilde{\omega}\in\Omega''}\frac{f(s|\tilde{\omega})}{f(s|\omega')}\mu(\tilde{\omega})} > \frac{\mu(\omega')}{\mu(\omega')+\varepsilon}.$$

Since for any $\varepsilon > 0$ we can find a positive-probability set of signals S' satisfying the above inequality, we conclude that for any $\varepsilon > 0$, $\Pr_\mu(s : \mu_s(\Omega') > 1-\varepsilon) > 0$.

We next prove that if ω' is distinguishable from Ω'', and Ω'' is finite, then the lemma's condition holds. Consider any μ uniformly distributed over $\{\omega'\}\cup\Omega''$. The distinguishability of ω' from Ω'' implies that for every $\varepsilon > 0$ there is a positive-probability set of signals S' such that $\forall s \in S'$ we have $\frac{\sum_{\tilde{\omega}\in\Omega''} f(s|\tilde{\omega})}{f(s|\omega')} < \varepsilon$, and so $\frac{f(s|\tilde{\omega})}{f(s|\omega')} < \varepsilon$ for every $\tilde{\omega} \in \Omega''$. *Q.E.D.*

Supplementary Appendices (Not For Publication)

SA.1. Belief Convergence

This section elaborates on Remark 6. One may expect the social belief to be eventually close to the stationary set with high probability: after all, when an agent's social belief is not close to the stationary set, her private information gives her a welfare improvement bounded away from zero; expanding observations should propagate these improvements, which implies (since utility is bounded) that they must eventually vanish. However, the following is a counterexample.[33]

Example SA.1. Consider binary states with a uniform prior, binary signals with symmetric precision (less than 1), and binary actions with simple utility. The network is as follows: agents 1 and 2 observe no one; for odd $n \geq 3$, agent n observes agent $n-2$; for even $n \geq 3$, agent n observes agent $n-1$ and agent 2. So there is expanding observations. In this network, the odd agents form an immediate-predecessor network and there is an equilibrium where a cascade along this subsequence starts from agent 3.

Now consider even agents. Consider the positive-probability event in which agents 1 and 2 take different actions. An even agent $n > 3$ observes agents $n-1$ and 2, which, given the equilibrium behavior of odd agents, is equivalent to observing agents 1 and 2. So the social belief of every even agent $n > 3$ equals the prior, which is bounded away from the stationary set. $\square$

The "problem" in Example SA.1 is that even though each of the even agents ($n > 3$) is getting a welfare improvement bounded away from zero, these improvements are not passed on to any future agents, and all future even agents continue to have social beliefs bounded away from the stationary set. In other words, expanding observations is not enough to validate the intuition described before the example. The following proposition identifies a reasonable condition on the network that is sufficient.

Proposition SA.1. *Assume there exist finitely many subsequences of agents $\{n_{k,j}\}_{k=1}^{N_j}$ ($j = 1, \ldots, J < \infty$, $1 \leq N_j \leq \infty$) such that agent $n_{k,j}$ observes $n_{k-1,j}$, and every agent in society is in at least one of the subsequences. Then, for all $\varepsilon > 0$, $\lim_{n\to\infty} \varphi_n(\mu_n \in S^\varepsilon) = 1$.*

The proposition's assumption encompasses canonical examples like the complete network and the k-immediate-predecessor networks (i.e., every agent observes the last k

[33] Absent expanding observations, there are trivial counterexamples using the empty network.

agents) for any $k \geq 1$. But it rules out any network in which infinite number of agents are not observed by any of their successors, which explains why it does not apply to Example SA.1.

Proof of Proposition SA.1. Along any subsequence j, $u(\varphi_{n_{k,j}}) \geq u(\varphi_{n_{k-1,j}}) + I(\varphi_{n_{k-1,j}})$ by the improvement principle, given that $n_{k-1,j}$ is observable to $n_{k,j}$. It follows that $\sum_{k=1}^{N_j} I(\varphi_{n_{k,j}}) \leq 2\overline{u}$. Hence, society's total improvement is bounded: $\sum_n I(\varphi_n) \leq 2\overline{u}J$.

Now fix any $\varepsilon, \delta > 0$. Consider $V_{\delta/2}$ defined in Lemma 3. The lemma established that $V_{\delta/2}$ is compact and $\varphi(\mu \notin V_{\delta/2}) < \delta/2, \forall \varphi \in \Phi^{BP}$. Since S^ε is open, $K := (S^\varepsilon)^c \cap V_{\delta/2}$ is compact. Next we argue $\mathbb{P}(\mu_n \in K\ i.o.) = 0$. Suppose, to contradiction, $\mathbb{P}(\mu_n \in K\ i.o.) > 0$. Then $\sum_n \mathbb{P}(\mu_n \in K) = \infty$ by the Borel-Cantelli lemma. Since K is compact and $I(\cdot) > 0$ on K, $I(\cdot)$ achieves its minimum in K at some $\underline{\mu} \in K$ with $I(\underline{\mu}) > 0$. So the total improvement is $\sum_n I(\varphi_n) \geq I(\underline{\mu}) \sum_n \mathbb{P}(\mu_n \in K) = \infty$, which contradicts $\sum_n I(\varphi_n) \leq 2\overline{u}J$.

Observe that $\mathbb{P}(\mu_n \in K\ i.o.) = 0$ implies $\varphi_n(\mu_n \in K) < \delta/2$ for all large n. Therefore, for all large n, $\varphi_n(\mu_n \in (S^\varepsilon)^c) \leq \varphi_n(\mu_n \in K) + \varphi_n(\mu_n \notin V_{\delta/2}) < \delta$. We conclude that for all $\varepsilon > 0$, $\lim_{n\to\infty} \varphi_n(\mu_n \in S^\varepsilon) = 1$. *Q.E.D.*

Remark 8. If $\Delta\Omega$ is compact (e.g., Ω itself is compact), we can replace $V_{\delta/2}$ in the proof with $\Delta\Omega$, so that $K = (S^\varepsilon)^c$. Then the argument in the proof's second paragraph shows that $\mathbb{P}(\mu_n \in (S^\varepsilon)^c\ i.o.) = 0$, i.e., the social belief converges to the stationary set almost surely rather than only in probability.

SA.2. ε-Excludability

This section elaborates on Remark 7. Say that for any $\varepsilon \in (0, 1/2)$ a set of states Ω' is ε-*distinguishable* from Ω'' if for any $\mu \in \Delta(\Omega' \cup \Omega'')$ with $\mu(\Omega') > \varepsilon$, there is a positive-measure set of signals S' such that $\mu(\Omega'|s) > 1-\varepsilon$ for all $s \in S'$. A utility function and an information structure jointly satisfy ε-*excludability* if Ω_{a_1,a_2} and Ω_{a_2,a_1} are ε-distinguishable from each other, for any pair of actions a_1, a_2. Note that ε-excludability implies ε'-excludability for all $\varepsilon' > \varepsilon$, and excludability is equivalent to ε-excludability for all $\varepsilon > 0$.

Proposition SA.2. *Let Ω be finite. For all $\varepsilon \in (0, 1/2)$, ε-excludability implies that in any equilibrium σ, $\liminf_n \mathbb{E}_{\sigma,\mu_0}[u_n] \geq u^*(\mu_0) - 2\overline{u}\frac{\varepsilon}{1-\varepsilon}|\Omega|$.*

Before proving Proposition SA.2, we give an example illustrating the result's use.

Example SA.2. There are three states, $\omega \in \{1, 2, 3\}$, SCD preferences, and Laplace information:

$$f(s|\omega) = \frac{1}{2b} \exp\left(-\frac{|s-\omega|}{b}\right),$$

where $b > 0$ is a scale parameter; a smaller b corresponds to more precise information.

It is straightforward to verify that no two states can be distinguished from each other.[34] Therefore, not every stationary belief has adequate knowledge (so long as preferences are nontrivial), and by Theorem 1 there is inadequate learning.

Nonetheless, we claim that ε-excludability holds for any ε such that $\varepsilon > \frac{1}{1+\exp(\frac{1}{2b})}$. To see this, observe that since the information structure has MLRP and preferences satisfy SCD, we can focus on ε-distinguishing state 3 from 2 (or, equally, 2 from 1).[35] When $\varepsilon > \frac{1}{1+\exp(\frac{1}{2b})}$, we have $\frac{\varepsilon}{1-\varepsilon}\exp(1/b) > \frac{1-\varepsilon}{\varepsilon}$, so there exist signals that move the prior $(0, 1-\varepsilon, \varepsilon)$ to a posterior of at least $1-\varepsilon$ on state 3, which implies ε-distinguishability of state 3 from 2.

Proposition SA.2 implies that in any equilibrium, $\liminf \mathbb{E}_{\sigma,\mu_0}[u_n] \geq u^*(\mu_0) - 6\overline{u}\exp(-\frac{1}{2b})$. This quantitative welfare bound yields, in particular, convergence to the full-information utility $u^*(\mu_0)$ as $b \to 0$. □

Proof of Proposition SA.2. Take any stationary belief μ, and let a be an optimal action at belief μ. For each state ω, take any $a_\omega \in c(\omega)$, and consider $\mu_\omega(\cdot) := \mu(\cdot|\{\omega\} \cup \Omega_{a,a_\omega})$. If $\mu_\omega(\omega) \leq \varepsilon$, then $\mu(\omega) \leq \varepsilon$, so $(u(a_\omega,\omega) - u(a,\omega))\mu(\omega) \leq 2\overline{u}\varepsilon$.

Consider the other case of $\mu_\omega(\omega) > \varepsilon$. For any $s \in S$, because $u(a,\omega') - u(a_\omega,\omega') \leq 0$ for each $\omega' \notin \Omega_{a,a_\omega}$, and μ is stationary,

$$\sum_{\omega' \in \{\omega\} \cup \Omega_{a,a_\omega}} (u(a,\omega') - u(a_\omega,\omega'))\mu(\omega'|s) \geq \sum_{\omega' \in \Omega} (u(a,\omega') - u(a_\omega,\omega'))\mu(\omega'|s) \geq 0.$$

Then,

$$\begin{aligned}(u(a_\omega,\omega) - u(a,\omega))\mu_\omega(\omega|s) &\leq \sum_{\omega' \in \Omega_{a,a_\omega}} (u(a,\omega') - u(a_\omega,\omega'))\mu_\omega(\omega'|s) \\ &\leq 2\overline{u}\left(\sum_{\omega' \in \Omega_{a,a_\omega}} \mu_\omega(\omega'|s)\right) = 2\overline{u}(1-\mu_\omega(\omega|s)).\end{aligned}$$

By ε-excludability, there exists a positive-measure set of signals S' such that, for any $s \in S'$, $\mu_\omega(\omega|s) > 1-\varepsilon$, which implies that $u(a_\omega,\omega) - u(a,\omega) \leq 2\overline{u}\frac{\varepsilon}{1-\varepsilon}$.

In either case ($\mu_\omega(\omega) \leq \varepsilon$ or $\mu_\omega(\omega) > \varepsilon$), we have $(u(a_\omega,\omega) - u(a,\omega))\mu(\omega) \leq 2\overline{u}\frac{\varepsilon}{1-\varepsilon}$. Since

[34] For any pair of states $\omega \neq \omega'$, and any signal s, the likelihood ratio $f(s|\omega)/f(s|\omega') \leq \exp(2/b)$.

[35] By MLRP, only arbitrarily large signals can distinguish a state from a lower state, and for large s the likelihood ratio $f(s|3)/f(s|2) < f(s|3)/f(s|1)$, so considering adjacent states is sufficient for ε-excludability.

Ω is finite,

$$\sum_{\omega\in\Omega}(u(a_\omega,\omega)-u(a,\omega))\mu(\omega)\leq 2\overline{u}\frac{\varepsilon}{1-\varepsilon}|\Omega|.$$

Namely, the utility gap $u^*(\mu)-u(\mu)\leq 2\overline{u}\frac{\varepsilon}{1-\varepsilon}|\Omega|$, for any stationary belief μ.

Finally, for any $\varphi\in\Phi^S$,

$$u^*(\mu_0)-u(\varphi)=\mathbb{E}_\varphi[u^*(\mu)-u(\mu)]\leq 2\overline{u}\frac{\varepsilon}{1-\varepsilon}|\Omega|.$$

By taking infimum of $u(\varphi)$ across $\varphi\in\Phi^S$, we obtain $u_*(\mu_0)\geq u^*(\mu_0)-2\overline{u}\frac{\varepsilon}{1-\varepsilon}|\Omega|$, and subsequently by invoking Theorem 3, we conclude that in any equilibrium σ, $\liminf_n \mathbb{E}_{\sigma,\mu_0}[u_n]\geq u^*(\mu_0)-2\overline{u}\frac{\varepsilon}{1-\varepsilon}|\Omega|$. *Q.E.D.*

SA.3. Learning at a Fixed Prior

For tractability, our discussion in this section assumes a complete network.

The issue. The definition of adequate learning we adopted in Section 2 requires that there is learning for all priors. In general, one may be interested in whether there is adequate learning at some given (full-support) prior.[36] Of course, our sufficient conditions—e.g., Theorem 2's excludability—remain sufficient, but fixing a prior raises the question of whether the conditions are necessary. With only two states, the distinction between some prior and all priors is immaterial: if adequate learning fails at any prior, then the only adequate-knowledge beliefs are those with certainty on some state, and there is an open ball of stationary beliefs around certainty on one of the states; hence, given any full-support prior, there cannot be a belief path that converges to certainty on that state, implying a failure of adequate learning at all full-support priors.

However, with multiple states, a failure of adequate learning at some prior does not imply an open ball of stationary beliefs around any adequate-knowledge belief. To illustrate, consider Figure SA.1. Action a^* is optimal at states 2 and 3 while $\underline{a}$ is optimal at state 1. Adequate learning fails when the prior is μ because μ, which has support $\{1,2\}$, is stationary but has inadequate knowledge.[37] Yet there is no open ball of stationary beliefs: no full-support belief is stationary because the optimal actions are distinct at the extreme states 1 and 3, and the extreme states are distinguishable from their complements. This raises the

[36] To be complete: there is *adequate learning at prior* μ_0 if for every equilibrium strategy profile σ, $\mathbb{E}_{\sigma,\mu_0}[u_n]\to u^*(\mu_0)$. Adequate (or inadequate) learning at a prior for a choice set is defined analogously.

[37] In this example, preferences have SCD. But, consistent with Proposition 1, DUB is violated because state 2 is not distinguishable from state 1 or state 3.

possibility that there is learning at some—or even all—full-support priors, with on-path sequences of beliefs (which necessarily have full support at every finite time) converging almost surely to adequate-knowledge beliefs without ever hitting any stationary belief (all of which have non-full-support).

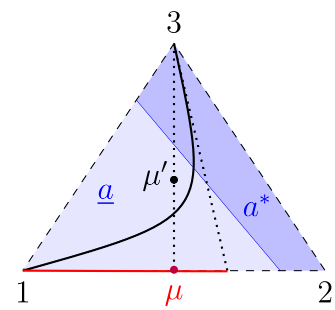

Figure SA.1: Preference regions among the actions $\underline{a}$ and a^* shaded in blues. Under belief μ', posteriors are given by the black curve, while under belief μ, posteriors are given by the red line.

Some partial analysis. While we are unable to characterize learning at a fixed prior in general, we provide some partial analysis below that we hope will be useful for future research. We focus on obtaining an analog of Theorem 2 for a fixed prior.

First, we provide a lemma (Lemma SA.1) that connects the existence of certain on-path histories to distinguishability. Second, we conjecture a result (Conjecture SA.1) and show in Proposition SA.3 that, if the result is true, it combines with the lemma to deliver a fixed-prior analog of Theorem 2. Third, we show that the conjecture is true in a class of problems (Claim SA.1).

Lemma SA.1. *Take an aribtrary $\omega^* \in \Omega$ and set of states $\Omega' \subseteq \Omega \backslash \{\omega^*\}$. State ω^* is distinguishable from Ω' if there exist an equilibrium under a full-support prior and a history of actions h^∞ such that $\mathbb{P}(h^\infty|\omega^*) = 0$, and $\mathbb{P}(h^\infty|\omega)$ is bounded away from 0 across $\omega \in \Omega'$.*

The lemma ties the asymptotic probabilities of on-path histories to the information structure of an individual agent. The formal proof of the lemma is provided at the end of this appendix, but to see the intuition, suppose that the relevant h^∞ in the hypothesis of the lemma is some eventual herd on some action $a \in A$, i.e., $h^\infty = \{h^m, a, a, a, , ...\}$ with h^m some finite subhistory. If h^∞ has 0 probability in state ω^*, it must be because in an infinite number of periods, agents have positive probability of obtaining signals which overturn the herd on a, i.e., result in them taking some other action than a. However, the probability of this history is positive for states in Ω'. This means that the probability of signals that overturn the herd must vanish over time at a fast enough rate in ω^*, but either not vanish or vanish at a slow enough rate in each $\omega \in \Omega'$. In particular, there must exist overturning signals whose probability gets arbitrarily large in state ω^* relative to those in every $\omega \in \Omega'$, which means ω^* is distinguishable from Ω'.

Conjecture SA.1. *Take any $a_1, a_2 \in A$, any full-support prior, and any equilibrium. If there is adequate learning at that prior, choice set $\{a_1, a_2\}$, and equilibrium,*[38] *then*

$$\exists h^\infty \text{ and } \varepsilon > 0 : \ \mathbb{P}(h^\infty|\omega) > \varepsilon, \forall \omega \in \Omega_{a_1,a_2}. \tag{SA.1}$$

The conjecture says that given any full-support prior, any binary choice set $\{a_1, a_2\}$, and any equilibrium in which there is adequate learning, we can find a single history that occurs with probability bounded away from 0 in all states in which a_1 is strictly preferred (and analogously, a different history for the states in which a_2 is strictly preferred). To appreciate the conjecture, let us focus for discussion on the case of finite states, nontrivial information, and nontrivial preferences. First note that if Ω_{a_1,a_2} is a singleton—as is the case with binary states—then it is straightforward that there is such a history, as there is a herd almost surely and every herd begins at some finite time. When Ω_{a_1,a_2} is not a singleton, given adequate learning, the same logic shows that for each state in Ω_{a_1,a_2}, there is a history that has positive probability in that state, namely one with a herd on a_1. But Conjecture SA.1 demands more: a single history that has positive probability in all states in Ω_{a_1,a_2}. Nonetheless, the conjecture seems intuitive: (up to tie-breaking issues) it would be surprising for every infinite history that has positive probability in some $\omega \in \Omega_{a_1,a_2}$ to have zero probability in some other $\omega' \in \Omega_{a_1,a_2}$, given that agents' have the same ordinal preferences over the binary actions in both ω and ω'. For instance, consider a fully-informative information structure and any nontrivial preferences. Clearly, given any choice set $\{a_1, a_2\}$, there are only two possible histories: either a_1 in every period or a_2 in every period. The former has probability 1 in each $\omega \in \Omega_{a_1,a_2}$, and the latter has probability 1 in each $\omega \in \Omega_{a_2,a_1}$ and so Conjecture SA.1 holds. Even though individuals' private information distinguishes states perfectly, the public history does not.

Proposition SA.3. *If Conjecture SA.1 is true, then not only does excludability imply adequate learning at every prior for every choice set, but moreover, if excludability fails, then there exists a choice set with inadequate learning at every full-support prior.*

Proof. That excludability implies adequate learning at every prior for every choice set is implied by Theorem 2, with no need to invoke Conjecture SA.1. So we only prove the second portion of the proposition, doing so by contraposition.

[38] That is, under the given prior μ_0 and equilibrium σ, $\mathbb{E}_{\sigma,\mu_0}[u_n] \to u^*(\mu_0)$.

To that end, assume that for every choice set, there is some full-support prior at which there is adequate learning in some equilibrium. For every binary choice set $\{a_1, a_2\}$, Conjecture SA.1 implies the existence of a history h^∞ satisfying (SA.1) at the full-support prior and equilibrium at which there is adequate learning. Since there is adequate learning, eventually all agents must be taking a_1 in h^∞, which implies that $\mathbb{P}(h^\infty|\omega^*) = 0$ for each $\omega^* \in \Omega_{a_2,a_1}$. Then, taking $\Omega' = \Omega_{a_1,a_2}$ in Lemma SA.1 yields that ω^* is distinguishable from Ω_{a_1,a_2}. Since a_1, a_2 and $\omega^* \in \Omega_{a_1,a_2}$ are arbitrary, there is excludability. *Q.E.D.*

We have not been able to establish Conjecture SA.1 in general. However, we are able to establish it when preferences satisfy SCD and the information structure satisfies the strict MLRP (assuming, only for convenience, that the state space is discrete):

Claim SA.1. *Assume $\Omega \subseteq \mathbb{Z}$. If preferences satisfy SCD and the information structure satisfies the strict MLRP, then Conjecture SA.1 is true.*

The proof is at the end of this appendix. Combining Claim SA.1 and Proposition SA.3, we see that under a complete network, the signal structure and preferences in Figure SA.1 entail inadequate learning at every full-support prior, such as μ' in the figure. Note that the figure's signal structure satisfies strict MLRP because the black curve in Figure SA.1 is concave vis-à-vis the 1–3 edge and approaches the 1 and 3 vertices.

Omitted Proofs

Proof of Lemma SA.1. Suppose not. Then there exist a belief $\mu \in \Delta(\Omega' \cup \{\omega^*\})$ with $\mu(\omega^*) > 0$ and a small $\varepsilon > 0$ such that for almost every signal s, the posterior $\mu(\omega^*|s) \leq 1 - \varepsilon$. By taking the conditional distribution of μ on Ω', call it $\tilde{\mu}$, and $z := \frac{\varepsilon\mu(\omega^*)}{1-\mu(\omega^*)} \in (0,1)$, we obtain for almost every s,

$$\int_{\Omega'} f(s|\omega)\,\mathrm{d}\tilde{\mu}(\omega) \geq z f(s|\omega^*). \tag{SA.2}$$

Suppose there exist an equilibrium σ under a full support prior and history h^∞ such that $\mathbb{P}(h^\infty|\omega^*) = 0$, and $\mathbb{P}(h^\infty|\omega)$ is bounded away from 0 across $\omega \in \Omega'$. Let a^n be the action taken by agent n along h^∞ and $A^{-n} := A\backslash\{a^n\}$. Let $\mathbb{P}(a^n|h^n,\omega) := \int_S \sigma(a^n|s,h^n)f(s|\omega)\,\mathrm{d}s$ be the probability that agent n plays action a^n when the state is ω and the sub-history is h^n. It holds that:

$$\begin{aligned}\sum_{n=1}^{\infty}\log(1 - z\mathbb{P}(A^{-n}|h^n,\omega^*)) &\geq \sum_{n=1}^{\infty}\log\left(1 - \int_{\Omega'}\mathbb{P}(A^{-n}|h^n,\omega)\,\mathrm{d}\tilde{\mu}(\omega)\right) \qquad \text{(using (SA.2))}\\ &= \sum_{n=1}^{\infty}\log\left(\int_{\Omega'}\mathbb{P}(a^n|h^n,\omega)\,\mathrm{d}\tilde{\mu}(\omega)\right)\end{aligned}$$

$$\geq \sum_{n=1}^{\infty} \int_{\Omega'} \log(\mathbb{P}(a^n|h^n,\omega))\,\mathrm{d}\tilde{\mu}(\omega) \qquad \text{(by Jensen's inequality)}$$

$$= \int_{\Omega'} \sum_{n=1}^{\infty} \log(\mathbb{P}(a^n|h^n,\omega))\,\mathrm{d}\tilde{\mu}(\omega) \qquad \text{(by Tonelli's theorem)}$$

$$= \int_{\Omega'} \log\left(\prod_{n=1}^{\infty} \mathbb{P}(a^n|h^n,\omega)\right) \mathrm{d}\tilde{\mu}(\omega)$$

$$> -\infty \qquad \text{(as } \log \mathbb{P}(h^\infty|\omega) \text{ is bounded across } \omega \in \Omega'). \qquad \text{(SA.3)}$$

Below we will invoke the fact that for arbitrary sequences (S_n) and (T_n) and constant $c > 0$, if $\lim_{n\to\infty} \frac{S_n}{T_n} = c > 0$ and $\sum_n S_n < \infty$, then $\sum_n T_n < \infty$.[39] Let $S_n = -\log(1 - z\mathbb{P}(A^{-n}|h^n,\omega^*))$ and $T_n = -\log(1-\mathbb{P}(A^{-n}|h^n,\omega^*))$. Note that $\lim_{n\to\infty} \frac{S_n}{T_n} = z \in (0,1)$ because $\lim_{x\to 0} \frac{\log(1-zx)}{\log(1-x)} = z$ and (SA.3) implies $\lim_{n\to\infty} \mathbb{P}(A^{-n}|h^n,\omega^*) = 0$. The aforementioned mathematical fact implies that

$$\sum_{n=1}^{\infty} \log(1 - \mathbb{P}(A^{-n}|h^n,\omega^*)) > -\infty.$$

As $\mathbb{P}(a^n|h^n,\omega^*) = 1 - \mathbb{P}(A^{-n}|h^n,\omega^*)$, it further follows that $\prod_{n=1}^{\infty} \mathbb{P}(a^n|h^n,\omega^*) > 0$, which contradicts $\mathbb{P}(h^\infty|\omega^*) = 0$. *Q.E.D.*

Proof of Claim SA.1. Take any information structure f with strict MLRP, and a utility function u that has SCD. Then, take an equilibrium σ under a full support prior μ and a binary choice set $\{a_1, a_2\}$. Since u has SCD, Ω_{a_1,a_2} and Ω_{a_2,a_1} are either an upper and lower set or the other way around. We consider the case that a_2 is preferred in higher states and a_1 is preferred in lower states. We omit an analogous proof for the other case.

We observe that $\mathbb{P}(a_1|h^n,\omega) := \int_S \sigma(a_1|h^n,s) f(s|\omega)\,\mathrm{d}s$, the probability that agent n plays action a_1 given any finite history h^n, decreases in ω. First, the probability $\sigma(a_1|h^n,s)$ decreases in s. The interim belief $\mu(\cdot|h^n)$ has full support, so strict MLRP of the information structure implies that $\forall s < s'$, the posterior $\mu(\cdot|h^n,s')$ strictly monotone likelihood-ratio dominates $\mu(\cdot|h^n,s)$. By Theorem 2 of Athey (2002),

$$D(s) := \sum_{\omega} (u(a_2,\omega) - u(a_1,\omega))\,\mu(\omega|h^n,s)$$

is strictly single crossing in s, i.e., $D(s) \geq 0 \implies D(s') > 0, \forall s' > s$. Hence, $\sigma(a_1|h^n,s)$ is

[39] For any $c' < c$ there exists N such that for all $n > N$, $S_n/T_n \geq c'$, or $T_n \leq S_n/c'$. So $\sum_n T_n \leq \sum_{n\leq N} T_n + \sum_{n>N} S_n/c' < \infty$.

decreasing in s. Since f satisfies strict MLRP, $\mathbb{P}(a_1|h^n, \omega)$ is decreasing in ω.

Next, suppose there is adequate learning. So for each state $\omega \in \Omega_{a_1,a_2}$, there is an infinite history with a herd on a_1, $h^\infty = (\ldots, a_1, a_1, \ldots)$ that occurs with positive probability in ω. In particular, if we let $\tilde{\omega} = \max \Omega_{a_1,a_2}$, then for any finite sub-history h^n of h^∞,

$$\forall \omega \leq \tilde{\omega} : \mathbb{P}(a_1|h^n, \omega) \geq \mathbb{P}(a_1|h^n, \tilde{\omega}) > 0,$$

and since h^∞ has positive probability at $\tilde{\omega} \in \Omega_{a_1,a_2}$, it follows that

$$\forall \omega \leq \tilde{\omega} : \prod_{n=1}^{\infty} \mathbb{P}(a_1|h^n, \omega) \geq \prod_{n=1}^{\infty} \mathbb{P}(a_1|h^n, \tilde{\omega}) > 0. \tag{SA.4}$$

This means $\mathbb{P}(h^\infty|\omega)$ is uniformly bounded away from 0 for $\{\omega : \omega \leq \tilde{\omega}\}$. Since $\Omega_{a_1,a_2} \subseteq \{\omega : \omega \leq \tilde{\omega}\}$, this establishes Conjecture SA.1. *Q.E.D.*