A Note on Undominated Bertrand Equilibria*

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Abstract

This note shows that the conventional outcome associated with Bertrand competition with homogenous products and different marginal costs is obtained in *every* Nash equilibrium in which firms use undominated strategies. This strengthens an existence result due to Blume (2003).

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The standard model of Bertrand competition with homogenous products and identical marginal costs has a unique Nash equilibrium: each firm prices at marginal cost. Unfortunately, these strategies are weakly dominated. Blume (2003) has shown that when marginal costs are *non-identical*, there is an equilibrium (in fact, more than one) in undominated strategies. These equilibria achieve the conventional market outcome of the lowest-cost firm serving the entire market at a price equal to the next-to-lowest marginal cost. A natural question in this case is whether *all* undominated equilibria have the same outcome in terms of market price and share. To my knowledge, this issue has not been settled in the literature.[1] This note provides an affirmative answer under mild assumptions.

Model. Consider the standard two-firm homogenous-products Bertrand pricing model with constant but non-identical marginal costs, $0 \leq c_1 < c_2$.[2] Market demand is given by a function $Q(p)$, where $Q : \mathbb{R} \to \mathbb{R}_+$. Denote $\Pi_i(p) := Q(p)[p - c_i]$ as the monopoly profit for firm i at price p. Firms choose prices simultaneously, denoted p_1 and p_2 respectively. If $p_i < p_j$, firm j's payoff is zero and firm i's payoff is $\Pi_i(p_i)$; if $p_1 = p_2$, each firm i gets $\frac{1}{2}\Pi_i(p_i)$.

I make the following five assumptions, all of which are satisfied in textbook examples:

(A0): There exists $\hat{\varepsilon} > 0$ such that $Q(\cdot)$ is Lipschitz continuous on $[c_2, c_2 + \hat{\varepsilon})$.

(A1): For any $p > c_2$ with $Q(p) > 0$, there exists some $i \in \{1, 2\}$ and $p' < p$ such that $\Pi_i(p') > \frac{1}{2}\Pi_i(p)$. In other words, firm i would rather charge the price $p' < p$ and sell to the whole market than split the market at p.[3]

(A2): For each $i \in \{1, 2\}$, $\max \Pi_i(p)$ exists and is finite. In other words, for each firm, there exists an optimal monopoly price.

(A3): For any $p < c_2$, $\Pi_1(p) < \Pi_1(c_2)$. In other words, as a monopolist, any price below c_2 would be strictly worse for firm 1 than pricing at c_2.

(A4): There exists $\tilde{\varepsilon} > 0$ such that $\Pi_2(p)$ is strictly increasing on $(c_2, c_2 + \tilde{\varepsilon})$. In other words, monopoly profits for firm 2 are strictly increasing just above price c_2.

Role of the Assumptions. (A0) and (A3) are used to construct an equilibrium where firm 1 serves the entire market at price c_2; (A3) and (A4) ensure that this equilibrium is in undominated

[1] Blume (2003) notes that lower market prices can be supported in equilibria with weakly dominated strategies.

[2] The analysis extends straightforwardly to more than two firms. What is important is that there is only one firm with the lowest cost.

[3] Note that this assumption is automatically satisfied if the demand function, $Q(p)$, is non-increasing.

strategies. (A1) and (A2) guarantee uniqueness of the desired market outcome; in particular, (A2) rules out the kinds of equilibria constructed by Baye and Morgan (1999).

Given that there are a continuum of pure strategies, say that a mixed strategy is *weakly undominated* if, viewed as a probability measure, it assigns zero probability to any (Borel) set of pure strategies that are all weakly dominated.

Proposition. *Any Nash equilibrium in weakly undominated strategies has the property that with probability one, firm 1 sells to the entire market at price* c_2*. Moreover, such an equilibrium exists.*

Proof. (A0) and (A3) imply existence of a Nash equilibrium (NE, hereafter) with the desired market price and share using Blume's (2003) construction: firm one plays a pure strategy of charging price c_2 and firm 2 mixes, say with a uniform distribution, on $[c_2, c_2 + \delta]$ for some small enough $\delta > 0$.[4] Firm 1's strategy is undominated because, by (A3), it is playing its unique best response to firm 2. Similarly, any $p_2 > c_2$ in firm 2's support would be the unique best response to firm 1 playing a uniform distribution on $[p_2, p_2 + \varepsilon(p_2)]$ for some small enough $\varepsilon(p_2) > 0$ (by (A0), (A4), and that δ can be chosen small enough); hence firm 2 is also playing an undominated strategy.

To prove the first statement of the Proposition, fix any undominated NE with strategies (σ_1, σ_2). For $i \in \{1, 2\}$, denote $\overline{p}_i := \sup[Supp[\sigma_i]]$ and $\underline{p}_i := \inf[Supp[\sigma_i]]$. By (A4), it is weakly dominated for firm 2 to charge any price less than or equal to c_2, so $\underline{p}_2 \geq c_2$ and σ_2 must put zero probability on c_2, which implies $\overline{p}_2 > c_2$. By (A3), any $p_1 < c_2$ is not a best response for firm 1, hence $\underline{p}_1 \geq c_2$. Therefore, it suffices to show that $\overline{p}_1 = c_2$.

Assume, to contradiction, that $\overline{p}_1 > c_2$. Notice that by (A4), firm 2 can get a positive expected profit by choosing some $p_2 = c_2 + \varepsilon$ for small enough $\varepsilon > 0$; hence σ_2 must put zero probability on prices that yield a zero expected profit against σ_1. This implies that $\overline{p}_2 \leq \overline{p}_1$. Similarly, since (A3) implies $\Pi_1(c_2) > 0$, it also follows that $\overline{p}_1 \leq \overline{p}_2$. Combined, we must have $\overline{p} := \overline{p}_1 = \overline{p}_2 > c_2$. Moreover, since charging any price above an optimal monopoly price is weakly dominated for a firm, (A2) implies that $\overline{p} < \infty$. There are now two cases to consider:

1) Suppose first that σ_2 puts positive probability on $\overline{p}$. Then $\overline{p}$ must yield firm 2 a positive expected profit, hence $Q(\overline{p}) > 0$ and σ_1 must also put positive probability on $\overline{p}$. But then (A1) implies that one of the two firms is not playing a best response.

2) Suppose next that σ_2 puts zero probability on $\overline{p}$. This implies that $\Pr_{\sigma_2}(p_2 \geq \overline{p} - \varepsilon) \to 0$ as $\varepsilon \downarrow 0$. But then, σ_1 is not a best response, because firm 1's expected profit from p_1 becomes

[4] Plainly, firm 2 is playing a best response to firm 1. It is routine to verify that (A0) ensures that $\delta > 0$ can be chosen small enough so that firm 1 does not want to raise price above c_2, while (A3) obviously implies that it does not want to decrease price below c_2.

arbitrarily small as $p_1 \uparrow \overline{p}$ (since (A2) implies that monopoly profits are bounded) whereas charging price c_2 gives firm 1 some strictly positive expected profit (by (A3)). *Q.E.D.*

References

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